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ON THE EXISTENCE OF INVARIANT SUBSPACES IN SPACES WITH INDEFINITE METRIC[1]

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ABSTRACT. Let P_1, P_2 be complementary projections in Hilbert space $\boldsymbol{H}$. Let U be a one-to-one and onto operator in $\boldsymbol{H}$ with $Q(Ux)=Q(x)$, where $Q(x)=\|P_1x\|^2-\|P_2x\|^2$. The sufficient condition is given for the unique existence of maximal subspace $\boldsymbol{L}$ invariant under all operators commuting with U, and such that $Q(x)\geqq 0$, $x\in\boldsymbol{L}$. The result was obtained in the course of attacking the problem proposed by Phillips [5] (see also [1]).

Let $\boldsymbol{H}$ be a (separable) complex Hilbert space, with the ordinary scalar product $(\cdot\,,\cdot)$, and with indefinite metric $Q(x,y)=(Jx,y)$ $(J=P_1-P_2;$ P_1, P_2 are two complementary orthogonal projections). Let $\mathscr{K}$ be the set of all bounded linear operators K from $\boldsymbol{H}$ to $P_2\boldsymbol{H}$ $(\equiv\boldsymbol{H}_2)$ with a bound $\leqq 1$ and such that $K\boldsymbol{H}_2=0$. Let $\mathscr{M}$ be the set of all maximal positive subspaces $\boldsymbol{M}$ in $\boldsymbol{H}$; M is said to be positive if $Q(x,x)\geqq 0$ for x in $\boldsymbol{M}$. Let $\mathscr{C}(T)$ be the set of all bounded linear operators that commute with T and its Q-adjoint operator T°; T° is defined by $Q(T^\circ x,y)=Q(x,Ty)$. U is called a Q-unitary operator if $Q(Ux,Uy)=Q(x,y)$ and if U is onto and one-to-one. Let $\boldsymbol{H}_j=P_j\boldsymbol{H}$ $(j=1,2)$. Let $U_{jk}=P_jUP_k$ $(j,k=1,2)$. Now our main result is the following:

THEOREM. *Let U be a Q-unitary operator. Suppose that there exist convex open sets Ω_1, Ω_2 with $\bar\Omega_1\cap\bar\Omega_2=\varnothing$ ($\bar\Omega$ means the closure of Ω) such that*

$$(*)\qquad \begin{aligned} &\{(U(1+K)x_1,x_1);\ \|x_1\|=1,\ x_1\in\boldsymbol{H}_1\}\subset\Omega_1,\\ &\{((1-K)Ux_2,x_2);\ \|x_2\|=1,\ x_2\in\boldsymbol{H}_2\}\subset\Omega_2,\end{aligned}$$

for all K in $\mathscr{K}$. Then there exists a unique maximal positive subspace invariant under T in $\mathscr{C}(U)$.

COROLLARY. *Suppose that A is a bounded Q-selfadjoint operator $(A=A^\circ)$ satisfying the condition $(*)$, given in the above theorem, with U replaced by A. Then there exists a unique maximal positive subspace invariant under T in $\mathscr{C}(A)$.*

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REMARK. The following example shows that there exist, generally, more than one invariant subspace in $\mathscr{M}$ when the condition (*) is violated. Let $\boldsymbol{H}$ be a two-dimensional complex euclidean space with element $x=\{x_1, x_2\}$. Define projections $P_1x=\{x_1, 0\}$, $P_2x=\{0, x_2\}$, and let A be an operator defined by $Ax=\{\lambda x_1+x_2, -x_1\}$ (λ:real). Then the A_λ satisfies all the conditions in the corollary if $|\lambda|>2$. Hence for $|\lambda|>2$ there exists a unique maximal positive subspace $\boldsymbol{M}_\lambda$ invariant under $\mathscr{C}(A_\lambda)$. On the other hand, by direct calculation, we can see that the set $\mathscr{M}_\lambda$ of all maximal positive subspaces invariant under $\mathscr{C}(A_\lambda)$ consists of $[\{x, (-\lambda+(\lambda^2-4)^{1/2})x/2\}; x\in\boldsymbol{C}]$ for $\lambda>2$; $[\{x, (-\lambda\pm(\lambda^2-4)^{1/2})x/2\}; x\in\boldsymbol{C}]$ for $|\lambda|\leqq 2$; $[\{x, (-\lambda-(\lambda^2-4)^{1/2})x/2\}; x\in\boldsymbol{C}]$ for $\lambda<-2$.

PROOF OF THEOREM. By the Q-unitarity of U,

$$\|x\|^2 + \|(U_{21} + zU_{22}K)x\|^2 \leqq |z|^2 \|Kx\|^2 + \|(U_{11} + zU_{12}K)x\|^2$$
$$\leqq |z|^2 \|x\|^2 + \|(U_{11} + zU_{12}K)x\|^2.$$

Hence,

$$(1)\qquad (1 - |z|^2) \|x\|^2 + \|(U_{21} + zU_{22}K)x\|^2 \leqq \|(U_{11} + zU_{12}K)x\|^2$$

for x in $\boldsymbol{H}_1$, K in $\mathscr{K}$ and $|z|\leqq 1$. Using (1), we shall show that for each K in $\mathscr{K}$ and $|z|<1$, $(U_{11}+zU_{12}K)^{-1}$ exists as a bounded operator on $\boldsymbol{H}_1$ with a bound

$$(2)\qquad \|(U_{11} + zU_{12}K)^{-1}\| \leqq (1 - |z|^2)^{-1/2}.$$

Indeed, $U_{11}+zU_{12}K$ is, by (1), one-to-one.

If y is orthogonal to $\boldsymbol{R}(U_{11}+zU_{12}K)$ ($\boldsymbol{R}(T)$=the range of T), then

$$0 = (y, (U_{11} + zU_{12}K)x) = Q(y, U(1 + zK)x)$$
$$= Q(U^\circ y, (1 + zK)x) = ([P_1U^\circ - \bar{z}K^*P_2U^\circ]y, x)$$

for all x in $\boldsymbol{H}_1$; K^* is the adjoint operator of K with respect to the ordinary scalar product (note that $K^*\boldsymbol{H}_1=0$, $\boldsymbol{R}(K^*)\subseteq\boldsymbol{H}_1$, and $\|K^*\|\leqq 1$). Hence $[P_1U^\circ-\bar{z}K^*P_2U^\circ]y=0$, from which it follows that $\|P_1U^\circ y\|\leqq |z|\,\|K^*\| \cdot \|P_2U^\circ y\|\leqq \|P_2U^\circ y\|$. Hence $\|y\|^2=Q(y, y)=Q(U^\circ y, U^\circ y)= \|P_1U^\circ y\|^2-\|P_2U^\circ y\|^2\leqq 0$. Here we used the Q-unitarity of U° implied by that of U. Hence $y=0$, i.e., the range of $U_{11}+zU_{12}K$ is dense in $\boldsymbol{H}_1$. Thus $U_{11}+zU_{12}K$ is, by (1), onto. The desired bound of $(U_{11}+zU_{12}K)$ follows from (1).

By Phillips [5], $\boldsymbol{M}\in\mathscr{M}$ if and only if $\boldsymbol{M}=\{x_1+Kx_1; x_1\in\boldsymbol{H}_1\}$ for some K in $\mathscr{K}$. Then it is easy to see that $\boldsymbol{M}$ is invariant under U if and only if K satisfies

$$(3)\qquad KU_{12}K + KU_{11} - U_{22}K - U_{21} = 0.$$

To find a solution K of (3) in $\mathscr{K}$, we consider instead the equation:

$$(4)\qquad zK(z)U_{12}K(z) + K(z)U_{11} - zU_{22}K(z) - U_{21} = 0$$

in $|z|<1$ and construct $K(z)$ from a sequence of holomorphic functions $\{K_j(z)\}$. We set

$$K_0(z) = 0 \quad \text{and} \quad K_{j+1}(z) = (U_{21} + zU_{22}K_j(z)) \cdot (U_{11} + zU_{12}K_j(z))^{-1}$$

i.e.,

$$K_{j+1}(z)(U_{11} + zU_{12}K_j(z)) = U_{21} + zU_{22}K_j(z). \tag{5}$$

Then the $K_j(z)$ has the properties:

(i) $K_j(z)\in\mathscr{K}$;

(ii) $K_j(z)$ is holomorphic in $|z|<1$;

(iii) for any compact set Δ in $|z|<1$ $\sup_\Delta\|K_j'(z)\|$ is uniformly bounded in j (K': derivative in z of K);

(iv) $K_j(z)$ converges uniformly to some bounded operator $\tilde{K}(z)$ on some small disk $|z|<\varepsilon$ in the operator norm.

Indeed, if $K_j(z)\in\mathscr{K}$, then we have, by (1),

$$\|K_{j+1}(U_{11}+ zU_{12}K_j)x\|^2 = \|(U_{21} + zU_{22}K_j)x\|^2 \leqq \|(U_{11} + zU_{12}K_j)x\|^2,$$

showing (i). Suppose that $K_j(z)$ is holomorphic in $|z|<1$. Then from (2) it is easy to see that $(U_{11}+zU_{12}K_j)^{-1}$ is holomorphic in $|z|<1$. Hence $(U_{21}+zU_{22}K_j(z))(U_{11}+zU_{12}K_j(z))^{-1}$ is holomorphic in z, showing (ii). (iii) is a direct consequence of (ii). Since

$$\begin{aligned} K_{j+1}(z) - K_j(z) = {} & zU_{22}(K_j - K_{j-1})(U_{11} + zU_{12}K_j)^{-1} \\ & + z^2(U_{21} + zU_{22}K_{j-1})(U_{11} + zU_{12}K_{j-1})^{-1} \\ & \cdot U_{12}(K_j - K_{j-1})(U_{11} + zU_{12}K_j)^{-1}, \end{aligned}$$

we have, by (2) and (i),

$$\|K_{j+1} - K_j\| \leqq \tfrac{1}{2} \|K_j - K_{j-1}\|$$

for sufficiently small $|z|\leqq\varepsilon$. Hence $K(z)$ converges to some $\tilde{K}(z)\in\mathscr{K}$ uniformly on $|z|\leqq\varepsilon$ in the operator norm. Thus we have (iv).

By (i), (ii), (iii), there exists a subsequence $K_{j'}(z)$ converging (in the weak topology) uniformly on any compact set in $|z|<1$ to some holomorphic (in $|z|<1$) operator-valued function $K(z)\in\mathscr{K}$. By (iv), $\tilde{K}(z)=K(z)$ in $|z|<\varepsilon$. Since $K_j(z)$ converges strongly to $\tilde{K}(z)$ in $|z|<\varepsilon$ and since K_j satisfies (5), $\tilde{K}(z)$, and so $K(z)$, satisfies (4) in $|z|<\varepsilon$. Since $K(z)$ is weakly, and so strongly, holomorphic in $|z|<1$, $K(z)$ must satisfy (4) in the whole unit disk $|z|<1$. Now we claim:

(v) $K'(z)$ is uniformly bounded for real z in $0<z<1$.

(vi) Equation (3) has at most one solution.

For the moment we assume that (v) and (vi) hold. Then $K(z)$ has a strong limit $K(1)\in\mathscr{K}$ for $z\to1$ $(0<z<1)$. Letting $z\to1$ in (4), we have

$$K(1)U_{12}K(1) + K(1)U_{11} - U_{22}K(1) - U_{21} = 0.$$

Thus the subspace $M=\{x_1+K(1)x_1;\ x_1\in H_1\}\in\mathscr{M}$ is invariant under U. Besides M is invariant also under T in $\mathscr{C}(U)$. Indeed, if T is in $\mathscr{C}(U)$, then $T+T^\circ$ is a Q-selfadjoint operator in $\mathscr{C}(U)$; note that $T^\circ\in\mathscr{C}(U)$ if $T\in\mathscr{C}(U)$. Hence the operator $V(t)$ defined by $V(t)=\exp[it(T+T^\circ)]$ is Q-unitary. Hence $V(t)$ maps M onto M. Since $UV(t)M=V(t)UM\subset V(t)M$, the space $V(t)M$ in $\mathscr{M}$ is invariant under U. By the uniqueness of the solution of (6) (see vi), which implies the uniqueness of invariant maximal positive subspace, we have $V(t)M=M$. Hence $(it)^{-1}[V(t)-I]M\subset M$. Letting $t\to 0$, we have $(T+T^\circ)M\subset M$. Similarly $i(T-T^\circ)M\subset M$. Thus $TM\subset M$.

PROOF OF (v). Since $\bar\Omega_1$, $\bar\Omega_2$ are closed, convex and disjoint sets, there exist θ $(-\pi/2\leqq\theta\leqq\pi/2)$, α_1, α_2 $(\alpha_1>\alpha_2)$ such that

$$\operatorname{Re}(e^{i\theta}\zeta_1) > \alpha_1 \quad\text{and}\quad \operatorname{Re}(e^{i\theta}\zeta_2) < \alpha_2, \tag{6}$$

or

$$\operatorname{Re}(e^{i\theta}\zeta_1) < \alpha_2 \quad\text{and}\quad \operatorname{Re}(e^{i\theta}\zeta_2) > \alpha_1 \tag{7}$$

for ζ_1 in $\bar\Omega_1$ and ζ_2 in $\bar\Omega_2$. We consider only the case (6); the case (7) can be treated similarly. Then by the assumption (*),

$$\operatorname{Re}(e^{i\theta}P_1U(1+K)x_1-\alpha_1x_1,x_1)\geqq 0,$$
$$\operatorname{Re}(e^{i\theta}P_2(1-K)Ux_2-\alpha_2x_2,x_2)\leqq 0,$$

for all $K\in\mathscr{K}$, $x_1\in H_1$ and $x_2\in H_2$. By this "dissipativity condition",

$$\begin{aligned}&\|\exp[-te^{i\theta}P_1U(1+zK)+t\alpha_1]P_1\|\leqq 1;\\ &\|\exp[te^{i\theta}zP_2(1-K)U-tz\alpha_2]P_2\|\leqq 1\end{aligned}\tag{8}$$

for all $t>0$, $K\in\mathscr{K}$ and $0<z\leqq 1$. Differentiation in z of (4) gives

$$K'(z)(U_{11}+zU_{12}K(z))+z(K(z)U_{12}-U_{22})K'(z)=B(z)$$

i.e.,

$$K'(z)P_1U(1+zK(z))P_1+zP_2(K(z)-1)UP_2K'(z)=B(z),$$

where $B(z)=-K(z)U_{12}K(z)+U_{22}K(z)$. Hence

$$K'(z)=-e^{-i\theta}\lim_{N\to\infty}\int_0^N e^{-(\alpha_1-z\alpha_2)t}\exp[te^{i\theta}zP_2(1-K(z))UP_2-\alpha_2zt]$$
$$\cdot P_2\cdot B(z)\cdot\exp[-te^{i\theta}\cdot P_1U(1+zK(z))P_1+\alpha_1t]P_1\,dt.$$

Hence, by (8),

$$\|K'(z)\|\leqq\|B(z)\|\cdot(\alpha_1-\alpha_2)^{-1}\leqq(\alpha_1-\alpha_2)^{-1}(\|U_{12}\|+\|U_{22}\|)$$

for $0<z<1$.

PROOF OF (vi). We consider only the case (6). Let K_1, K_2 be solutions of (3) in $\mathscr{K}$. Then $K_0=K_1-K_2$ satisfies $K_0(U_{12}K_2+U_{11})=(U_{22}-K_1U_{12})K_0$.

Hence we have

$$K_0 = -e^{-i\theta}\int_0^N \frac{d}{dt}\{\exp[te^{i\theta}P_2(1-K_1)UP_2]P_2K_0 \cdot \exp[-te^{i\theta}P_1U(1+K_2)]P_1\}\,dt$$
$$+ e^{-i\theta}e^{-(\alpha_1-\alpha_2)N}\exp[Ne^{i\theta}P_2(1-K_1)UP_2-\alpha_2N]$$
$$\cdot P_2K_0\exp[-Ne^{i\theta}P_1U(1+K_2)P_1+\alpha_1N]P_1$$
$$= e^{-i\theta-(\alpha_1-\alpha_2)N}\exp(\cdots)K_0\exp(\cdots).$$

Letting $N\to\infty$, we have, by (9), $K_0=0$. Thus the proof is completed.

PROOF OF COROLLARY. The Q-selfadjoint operator A defines the one-parameter family of group $U(t)=e^{itA}$ of Q-unitary operators. Then we have

$$U(t) = I + (itA) + \frac{1}{2!}(itA)^2 + \cdots.$$

Hence there exists a t_0 such that, for each $0<t<t_0$, $U(t)$ satisfies the condition (*). By the theorem, for each $0<t<t_0$ there exists the unique subspace $\boldsymbol{M}$ in $\mathscr{M}$ invariant under $\mathscr{C}(U(t))$. Since $\mathscr{C}(A)\subset\mathscr{C}(U(t))$, $\boldsymbol{M}$ is invariant under $\mathscr{C}(A)$. Conversely if $\boldsymbol{M}$ ($\in\mathscr{M}$) is invariant under $\mathscr{C}(A)$, then it is also invariant under $\mathscr{C}(U(t))$ by the above expansion. Hence the $\boldsymbol{M}$ is the unique subspace in $\mathscr{M}$ invariant under $\mathscr{C}(A)$.

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