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CONSTRUCTIONS OF DISJOINT STEINER TRIPLE SYSTEMS

JEAN DOYEN[1]

ABSTRACT. Let $D^*(v)$ denote the maximum number of pairwise disjoint and isomorphic Steiner triple systems of order v. The main result of this paper is a lower bound for $D^*(v)$, namely $D^*(6t+3) \geqq 4t-1$ or $4t+1$ according as $2t+1$ is or is not divisible by 3, and $D^*(6t+1) \geqq t/2$ or t according as t is even or odd. Some other related problems are studied or proposed for study.

1. **Introduction and historical note.** Given a finite nonempty set S of v elements (called *points*), a *Steiner triple system* of *order* v on S is a collection $\mathscr{S}$ of subsets of S (called *lines*) such that every line has exactly 3 points and every pair of points is contained in one and only one line. Any Steiner triple system is also a balanced incomplete block design with parameters v, $k=3$ and $\lambda=1$ (see for instance Hall [**10**, Chapter 15]).

Kirkman [**11**] proved in 1847 that a necessary and sufficient condition for the existence of a Steiner triple system (briefly STS) of order v is $v \equiv 1$ or 3 (mod 6). An STS of order v is sometimes denoted simply by $S(v)$.

Let $\mathscr{S}$ and $\mathscr{S}'$ be two STS on the same set S of points. $\mathscr{S}$ and $\mathscr{S}'$ are called *disjoint* if $\mathscr{S} \cap \mathscr{S}' = \varnothing$, that is if they have no line in common. According to [**8**], the construction of disjoint STS might be useful in the design of certain statistical experiments.

Let us denote by $D(v)$ the maximum number of pairwise disjoint $S(v)$ that can be constructed on a set S of v points. As S contains $v(v-1)(v-2)/6$ subsets of cardinality 3 and as any $S(v)$ contains exactly $v(v-1)/6$ lines, we have $D(v) \leqq v-2$, except of course if $v=1$. We shall denote by $D^*(v)$ the maximum number of pairwise disjoint and isomorphic $S(v)$ that can be constructed on S. Obviously, $1 \leqq D^*(v) \leqq D(v)$.

It is clear that

$$D^*(1) = D(1) = 1 \quad \text{and} \quad D^*(3) = D(3) = 1.$$

Cayley [**6**] proved in 1850 that $D^*(7) = D(7) = 2$. The following collections

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[1] Aspirant du Fonds National belge de la Recherche Scientifique.

of subsets of the set $\{a, b, c, d, e, f, g\}$ form two disjoint $S(7)$:

$$\mathscr{S} = \{\{a, b, c\}, \{c, d, e\}, \{e, f, a\}, \{a, d, g\}, \{b, e, g\}, \{c, f, g\}, \{b, d, f\}\},$$
$$\mathscr{S}' = \{\{a, b, e\}, \{b, c, f\}, \{c, d, a\}, \{d, e, f\}, \{f, g, a\}, \{b, d, g\}, \{c, e, g\}\}.$$

The same year (1850), Kirkman **[12]** proved that $D^*(9)=D(9)=7$. This result was "discovered" again by Sylvester (**[18]**, **[19]**) in 1861, Walecki in 1883 (see Lucas **[14**, 161–197**]**), Bays **[4]** in 1917 and finally Emch **[9]** in 1929 (for more historical details, see Ahrens **[1**, 110–113**]**). The simplest description of 7 pairwise disjoint $S(9)$ on the set $\{a, b, c, d, e, f, g, h, i\}$ is given by the following square arrays

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \qquad \begin{matrix} a & b & d \\ e & f & g \\ h & i & c \end{matrix} \qquad \begin{matrix} d & e & g \\ h & i & a \\ b & c & f \end{matrix} \qquad \begin{matrix} g & h & a \\ b & c & d \\ e & f & i \end{matrix}$$
$$\begin{matrix} g & a & b \\ c & d & e \\ f & h & i \end{matrix} \qquad \begin{matrix} a & d & e \\ f & g & h \\ i & b & c \end{matrix} \qquad \begin{matrix} d & g & h \\ i & a & b \\ c & e & f \end{matrix}$$

The 12 lines of each system are simply the 3 rows, the 3 columns and the 6 products involved in the expansion of the "determinant" of each array.

The other values of $D^*(v)$ and $D(v)$ are unknown. Besides a few isolated lower bounds such as $D(13)\geqq 3$, $D(15)\geqq 2$ (Kirkman **[13]**), $D(31)\geqq 6$ (Assmus and Mattson (**[2]**, **[3]**)), the only known general results are $D^*(2^n-1)\geqq 2$ for every odd integer $n\geqq 3$ (Assmus and Mattson **[2]**) and $D^*(6t+1)\geqq 2$ for every $t>0$: indeed, as was shown by Rosa **[16]** and Di Paola **[7]**, it is not difficult to construct two disjoint and isomorphic *cyclic* STS of order $6t+1$ (an $S(v)$ is called *cyclic* if one of its automorphisms is a cycle of length v).

In 1917, Bays **[4]** conjectured that $D(v)\geqq(v-1)/2$ for every $v\equiv 1$ or 3 (mod 6), $v>7$. Our first theorem shows that this conjecture is true for every $v\equiv 3$ (mod 6), even if $D(v)$ is replaced by $D^*(v)$.

2. A lower bound for $D^*(v)$.

THEOREM 1. *For every nonnegative integer t,*

$$D^*(6t+3) \geqq 4t+1 \quad \textit{if}\ 2t+1 \not\equiv 0 \pmod 3,$$

and

$$D^*(6t+3) \geqq 4t-1 \quad \textit{if}\ 2t+1 \equiv 0 \pmod 3.$$

PROOF. Let $G=\{1, a, a^2, \cdots, a^{2t}\}$ be a multiplicative cyclic group of order $2t+1$ and let us consider the Cartesian product $S=G\times\{0, 1, 2\}$. For every $e\in\{0, 1, 2\}$, the subset $G\times\{e\}$ of S will be denoted by G_e and any element (x, e) of G_e by $(x)_e$ or, when there is no danger of confusion, simply by x_e.

The set $\mathcal{S}$ consisting of (i) all subsets $\{x_0, x_1, x_2\}$ of S for any $x \in G$, (ii) all subsets $\{x_0, y_0, z_1\}$, $\{x_1, y_1, z_2\}$, $\{x_2, y_2, z_0\}$ of S for any $x, y, z \in G$, where $x \neq y$ and $xy = z^2$, is easily verified to be an STS of order $6t+3$; this construction is essentially due to Bose [5].

(a) Let $\varphi_0, \varphi_1, \cdots, \varphi_{2t}$ be $2t+1$ permutations of the set S defined as follows: for every $x \in G$ and every $i=0, 1, \cdots, 2t$,

$$\varphi_i(x_0) = x_0, \quad \varphi_i(x_1) = (a^i x)_1, \quad \varphi_i(x_2) = (a^{2t-i}x)_2.$$

Let $\mathcal{S}_i$ be the STS whose lines are the images of the lines of $\mathcal{S}$ by the permutation φ_i. The systems $\mathcal{S}_0, \mathcal{S}_1, \cdots, \mathcal{S}_{2t}$ obtained in this way are clearly isomorphic; we are going to prove that they are also pairwise disjoint.

Let $\mathcal{S}_i$, $\mathcal{S}_j$ be any two of the above systems, with $i \neq j$ $(i, j=0, 1, \cdots, 2t)$.

Any line of $\mathcal{S}_i$ having a point in G_0, G_1 and G_2 is of the form $\{x_0, (a^i x)_1, (a^{2t-i}x)_2\}$; in $\mathcal{S}_j$, such a line is $\{x'_0, (a^j x')_1, (a^{2t-j}x')_2\}$. If these lines coincide, we must have

$$x = x', \quad a^i x = a^j x', \quad a^{2t-i}x = a^{2t-j}x'$$

which implies $a^i = a^j$, a contradiction since $i \neq j$.

Any line of $\mathcal{S}_i$ having two points in G_0 is of the form $\{x_0, y_0, (a^i z)_1\}$ where $z^2 = xy$; in $\mathcal{S}_j$, such a line is $\{x'_0, y'_0, (a^j z')_1\}$ where $z'^2 = x'y'$. If they coincide, we have either

$$\begin{aligned} x &= x', & \qquad x &= y', \\ y &= y', \quad \text{or} & y &= x', \\ a^i z &= a^j z', & a^i z &= a^j z'. \end{aligned}$$

As G is abelian of odd order, we find in both cases $a^i = a^j$, a contradiction.

By similar straightforward computations, one can easily check that no line of $\mathcal{S}_i$ having two points in G_1 or G_2 can coincide with a line of $\mathcal{S}_j$ and therefore $\mathcal{S}_i$ and $\mathcal{S}_j$ are disjoint.

(b) Let σ be the permutation of S defined by $\sigma(x_0) = x_2$, $\sigma(x_1) = x_1$ and $\sigma(x_2) = x_0$ for every $x \in G$. Let $\mathcal{S}'_i$ $(i=0, 1, \cdots, 2t)$ be the STS whose lines are the images of the lines of $\mathcal{S}_i$ by the permutation σ. It is clear that $\mathcal{S}_0, \mathcal{S}_1, \cdots, \mathcal{S}_{2t}, \mathcal{S}'_0, \mathcal{S}'_1, \cdots, \mathcal{S}'_{2t}$ are isomorphic and that $\mathcal{S}'_0, \mathcal{S}'_1, \cdots, \mathcal{S}'_{2t}$ are pairwise disjoint.

If a system $\mathcal{S}'_i$ has a line in common with a system $\mathcal{S}_j$, this line must necessarily have a point in G_0, G_1 and G_2. In $\mathcal{S}_j$, any such line is of the form $\{x_0, (a^j x)_1, (a^{2t-j}x)_2\}$; in $\mathcal{S}'_i$, it is $\{(a^{2t-i}x')_0, (a^i x')_1, x'_2\}$. If these lines coincide, we have

$$x = a^{2t-i}x', \qquad a^j x = a^i x', \qquad a^{2t-j}x = x',$$

which gives $a^{2t-2i+j}=1$ and $a^{4t-i-j}=1$, that is $a^{3i}=a^{6t}$. Let us exclude the systems $\mathscr{S}'_i$ which may have a line in common with one of the systems $\mathscr{S}_0$, $\mathscr{S}_1, \cdots, \mathscr{S}_{2t}$. As the number of distinct cube roots of a^{6t} in the group G is three or one according as the order of G is or is not divisible by 3, the number of excluded systems will be three or one, and the theorem follows immediately.

COROLLARY 1. *$D^*(v) \geqq 2$ for every $v \geqq 7$, $v \equiv 1$ or* 3 (mod 6).

This follows from Theorem 1 and from Rosa's result mentioned in the introduction.

COROLLARY 2. *For every $v \geqq 7$, $v \equiv 1$ or* 3 (mod 6), *there exists a balanced incomplete block design with parameters v, $k=3$ and $\lambda=2$, all of whose blocks are distinct* (*compare with Theorem* 15.4.4 *in Hall* **[10]**).

THEOREM 2. *For every nonnegative integer t,*

$$D^*(6t+1) \geqq t/2 \quad \text{if } t \equiv 0 \pmod 2,$$

and

$$D^*(6t+1) \geqq t \quad \text{if } t \not\equiv 0 \pmod 2.$$

PROOF. Let $G=\{1, a, a^2, \cdots, a^{2t-1}\}$ be a multiplicative cyclic group of order $2t$ and let us consider the set $S=(G\times\{0, 1, 2\})\cup\{\infty\}$ of cardinality $6t+1$, where ∞ is a new symbol. For every $e\in\{0, 1, 2\}$, the element (x, e) of the subset $G\times(e\}$ will be denoted by $(x)_e$ or, when there is no danger of confusion, by x_e. Finally, let $L=\{1, a, a^2, \cdots, a^{t-1}\}$, $R=\{a^t, a^{t+1}, \cdots, a^{2t-1}\}$ and let $\mathscr{S}$ be the set consisting of

(i) all subsets $\{x_0, x_1, x_2\}$ of S for any $x\in L$,

(ii) all subsets $\{\infty, x_0, (a^t x)_2\}$, $\{\infty, x_1, (a^t x)_0\}$, $\{\infty, x_2, (a^t x)_1\}$ of S for any $x\in L$,

(iii) all subsets $\{x_0, y_0, z_1\}$, $\{x_1, y_1, z_2\}$, $\{x_2, y_2, z_0\}$ of S for any x, $y\in G$ with $x\neq y$ and

(1) $z\in L$ and $z^2=xy$ if $xy=a^{2j}$,

(2) $z\in R$ and $az^2=xy$ if $xy=a^{2j+1}$.

It is not difficult to verify that $\mathscr{S}$ is an STS of order $6t+1$; this construction is due to Skolem **[17]**.

Let $\varphi_0, \varphi_1, \cdots, \varphi_{t-1}$ be t permutations of the set S defined as follows; for every $x\in G$ and every $i=0, 1, \cdots, t-1$,

$$\varphi_i(x_0) = x_0, \qquad \varphi_i(x_1) = (a^i x)_1,$$
$$\varphi_i(x_2) = (a^{2t-1-i}x)_2, \quad \text{and} \quad \varphi_i(\infty) = \infty.$$

Let $\mathscr{S}_i$ be the STS whose lines are the images of the lines of $\mathscr{S}$ by the permutation φ_i. The systems $\mathscr{S}_0, \mathscr{S}_1, \cdots, \mathscr{S}_{t-1}$ are clearly isomorphic. Moreover a proof similar to that of the preceding theorem shows that $\mathscr{S}_0, \mathscr{S}_1, \cdots, \mathscr{S}_{t/2-1}$ are pairwise disjoint if t is even and that $\mathscr{S}_0, \mathscr{S}_1, \cdots, \mathscr{S}_t$ are pairwise disjoint if t is odd. The computations involved in this proof being quite straightforward, they will not be reproduced here.

3. **A lower bound for** $D(v)$. The two preceding theorems obviously give a lower bound for $D(v)$, since $D^*(v) \leqq D(v)$. We want to prove now that this lower bound is not best possible and can be improved in certain cases. For instance, Theorem 2 gives $D(19) \geqq 3$; our next result will show that $D(19) \geqq 9$.

THEOREM 3. *For every* $v \geqq 7$ *with* $v \equiv 1$ *or* 3 (mod 6),

$$D(2v+1) \geqq D(v)+2.$$

PROOF. Let $D(v)=d$ and let S, S' be two disjoint sets of cardinality v. We shall denote by $\mathscr{S}_1, \mathscr{S}_2, \cdots, \mathscr{S}_d$ d pairwise disjoint STS of order v on the set S, and by $\mathscr{S}'_{d+1}, \mathscr{S}'_{d+2}$ two disjoint STS of order v on the set S' (the existence of at least two such systems follows from Corollary 1 and our hypothesis $v \geqq 7$).

Let α be any permutation of S consisting of a single cycle of length v and let φ be any bijection from S' onto S. Finally let us consider the set $T=S \cup S' \cup \{\infty\}$ of cardinality $2v+1$, where ∞ is a new symbol.

We are going to construct $d+2$ Steiner triple systems $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_{d+2}$ on the set T. For every $i=1, 2, \cdots, d$, the lines of $\mathscr{T}_i$ will be

(i) all lines of $\mathscr{S}_i$,

(ii) all subsets $\{\infty, x, \alpha^{i-1}(\varphi(x))\}$ of T, where x is any point of S',

(iii) all subsets $\{x, y, \alpha^{i-1}(\varphi(z))\}$, $\{x, \alpha^{i-1}(\varphi(y)), z\}$, $\{\alpha^{i-1}(\varphi(x)), y, z\}$ of T, where $\{x, y, z\}$ is any line of $\mathscr{S}'_{d+1}$.

For $i=d+1$ or $d+2$, the lines of $\mathscr{T}_i$ will be

(i) all lines of $\mathscr{S}'_i$,

(ii) all subsets $\{\infty, x, \alpha^{i-1}(\varphi(x))\}$ of T, where x is any point of S',

(iii) all subsets $\{x, \alpha^{i-1}(\varphi(y)), \alpha^{i-1}(\varphi(z))\}$, $\{\alpha^{i-1}(\varphi(x)), y, \alpha^{i-1}(\varphi(z))\}$, $\{\alpha^{i-1}(\varphi(x)), \alpha^{i-1}(\varphi(y)), z\}$ of T, where $\{x, y, z\}$ is any line of $\mathscr{S}'_{d+1}$.

It is easy to check that each $\mathscr{T}_i$ is an $S(2v+1)$ and that $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_{d+2}$ are pairwise disjoint. This verification is rather tedious and will be omitted here.

COROLLARY 3. *For every odd integer* $t \geqq 1$,

$$D(6t+1) \geqq 2t-1.$$

PROOF. If $t=1$, the result is trivial. If $t=2t'+1\geqq 3$, then $6t+1=2(6t'+3)+1$ and so, by Theorems 3 and 2,

$$D(6t+1) \geqq D(6t'+3)+2 \geqq 4t'+1 = 2t-1.$$

4. **Disjoint and isomorphic cyclic Steiner triple systems.** Let us denote by $D_c^*(v)$ the maximum number of pairwise disjoint and isomorphic cyclic STS of order v. So for instance $D_c^*(1)=D_c^*(3)=1$, $D_c^*(7)=2$ and $D_c^*(9)=0$.

The following result is essentially due to Rosa [16].

THEOREM 4. *For every positive integer t,*

$$D_c^*(6t+1) \geqq 2.$$

PROOF. Peltesohn [15] has established the existence of a cyclic $S(v)$ for every $v\equiv 1$ or 3 (mod 6), except $v=9$. Let $\mathcal{S}$ be a cyclic $S(6t+1)$ constructed on the set $S=\{0, 1, \cdots, 6t\}$ in such a way that the permutation $\alpha=(0, 1, \cdots, 6t)$ be an automorphism of $\mathcal{S}$. The distance d_{ij} of the points i and j ($i, j=0, 1, \cdots, 6t$) will be defined as

$$d_{ij} = \min\{|i-j|, 6t+1-|i-j|\}.$$

For every line $\{i, j, k\}$ of $\mathcal{S}$, the 3 distances d_{ij}, d_{jk}, d_{ki} are distinct. Indeed, suppose for instance that $d_{ij}=d_{jk}$ and let α_{ij} be the power of α mapping i onto j. As $d_{ij}=d_{jk}$, α_{ij} maps j onto k and therefore also k onto i, otherwise the points j and k would belong to two distinct lines of $\mathcal{S}$. We conclude that $d_{ij}=d_{jk}=d_{ki}=(6t+1)/3$, which is clearly impossible.

Let $\mathcal{S}'$ be the STS whose lines are the images of the lines of $\mathcal{S}$ by the involution $\sigma=(0)(1, 6t)(2, 6t-1)\cdots(i, 6t+1-i)\cdots(3t, 3t+1)$. $\mathcal{S}'$ is isomorphic to $\mathcal{S}$. Moreover $\mathcal{S}$ and $\mathcal{S}'$ are disjoint. Indeed, let $\{i, j, k\}$ (resp. $\{i, j, k'\}$) be the line of $\mathcal{S}$ (resp. $\mathcal{S}'$) containing the points i and j; it is easily seen that $d_{ik}=d_{jk'}$. Therefore these two lines are distinct, otherwise $k=k'$ and $d_{ik}=d_{jk}$, a contradiction.

REMARK. If $\mathcal{S}$ is any cyclic $S(6t+3)$ constructed on the set $S=\{0, 1, \cdots, 6t+2\}$ and admitting the permutation $\alpha=(0, 1, \cdots, 6t+2)$ as an automorphism, then $\mathcal{S}$ necessarily contains the lines $\{i, 2t+1+i, 4t+2+i\}$ for every $i=0, \cdots, 2t$, and so $\mathcal{S}$ and its image $\mathcal{S}'$ by the permutation $\sigma=(0)(1, 6t+2)(2, 6t+1)\cdots(3t+1, 3t+2)$ are never disjoint.

THEOREM 5. *For every positive integer $t\not\equiv 1$ (mod 3),*

$$D_c^*(6t+3) \geqq 4t+1.$$

PROOF. Let $\mathcal{S}$ be the $S(6t+3)$ constructed in the proof of Theorem 1. The permutations π_1 and π_2 of S such that for every $x\in G$

$$\pi_1(x_0)=x_1, \qquad \pi_1(x_1)=x_2, \qquad \pi_1(x_2)=x_0,$$
$$\pi_2(x_i)=(ax)_i \qquad (i=0, 1, 2),$$

are clearly two automorphisms of $\mathcal{S}$. Moreover if $2t+1 \not\equiv 0 \pmod 3$, the permutation $\pi_1\pi_2$ consists of a single cycle of length $6t+3$ and $\mathcal{S}$ is a cyclic STS. The above inequality is then an immediate consequence of Theorem 1.

5. **Open problems.** (1) Given a Steiner triple system $\mathcal{S}$ of order $v \geqq 7$ on a set S of cardinality v, is there always another Steiner triple system $\mathcal{S}'$ isomorphic to $\mathcal{S}$ and disjoint from $\mathcal{S}$? In other words, is there always a permutation α of S such that the image of any line of $\mathcal{S}$ by α is never a line of $\mathcal{S}$?

(2) Is it true that $D_c^*(6t+3) \geqq 2$ for every $t \geqq 2$?

(3) The lower bounds for $D(v)$ given in this paper can certainly be improved. It is tempting to conjecture that $D(v)=v-2$ for every $v \geqq 9$, $v \equiv 1$ or 3 (mod 6).

(4) Given an integer n such that $0 \leqq n \leqq v(v-1)/6$, let us denote by $D(v, n)$ the maximum number of STS of order v that can be constructed on a set of cardinality v in such a way that any two of them have exactly n lines in common, these n lines being moreover in each of the $D(v)$ systems. It is an easy exercise to check that $D(7, 0)=D(7)=2$, $D(7, 1)=3$, $D(7, 2)=0$, $D(7, 3)=2$, $D(7, 4)=D(7, 5)=0$, $D(7, 7)=\infty$. Kirkman **[12]** proved in 1850 that $D(15, 5) \geqq 15$, but almost nothing is known in general about the function $D(v, n)$. For example, is it true that $D(v, 1) \geqq 2$ for every $v \geqq 3$, $v \equiv 1$ or 3 (mod 6)?

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Department of Mathematics, University of Brussels, 1050 Brussels, Belgium