THE CONVERSE TO A THEOREM OF SHARP
ON GORENSTEIN MODULES

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Abstract. Let $A$ be a commutative local Noetherian ring with identity of Krull dimension $n$, $m$ its maximal ideal. Sharp has proved that if $A$ is Cohen-Macaulay and a homomorphic image of a Gorenstein local ring, then $A$ has a Gorenstein module $M$ with $\dim_{A/m}\text{Ext}^n(A/m, M)=1$. The aim of this note is to prove the converse to this theorem.

Throughout this note $A$ will denote a commutative local Noetherian ring with identity; $m$ will denote its maximal ideal. The concept of a Gorenstein module was introduced by Sharp in [8].

(1) Definition. A nonzero finitely generated $A$-module $M$ is called Gorenstein if the Cousin complex [7] provides a minimal injective resolution of $M$.

Sharp obtained various characterizations and properties of Gorenstein modules in [8]. In particular, he showed that for there to exist a Gorenstein $A$-module it is necessary that $A$ be Cohen-Macaulay [8, (3.9)], and he showed that a Gorenstein $A$-module has zero annihilator [8, (4.12)]. It follows that (in the notation of [8]) if $M$ is a Gorenstein module, then $\mu^i(m, M)=0$ if and only if $i\neq K$-$\text{dim } A$ [8, (3.11)]. We define the rank of the Gorenstein $A$-module $M$ to be $\mu^n(m, M)$, where $n=K$-$\text{dim } A$. In [9] Sharp proved the following

(2) Theorem. If $A$ is Cohen-Macaulay and a quotient of a Gorenstein local ring, then $A$ has a Gorenstein module of rank 1 [9, (3.1)].

The aim of this note is to prove the following converse:

(3) Theorem. If $A$ has a Gorenstein module $M$ of rank 1, then $A$ is a quotient of a Gorenstein local ring.
The notation will be the same as that in [8], with the following exception: if \( M \) is a nonzero finitely generated \( A \)-module, the notation \( \text{depth}_A M \) will be used instead of \( \text{codim}_A M \).

(4) The principle of idealization, introduced by Nagata (see [6, p. 2]) will be our tool. From the ring \( A \) and an \( A \)-module \( N \), we obtain a structure of a commutative ring with identity on the Cartesian product set \( A \times N \). Addition is "componentwise", and multiplication is given by
\[
(a_1, n_1) \cdot (a_2, n_2) = (a_1a_2, a_1n_2 + n_1a_2).
\]
\( A \) is then a quotient ring of \( A \times N \), for \( A \times N/0 \times N \cong A \). Since \( 0 \times N \) is nilpotent, all prime ideals in \( A \times N \) are of the form \( p \times N \) for some prime ideal \( p \) of \( A \). Hence \( A \times N \) is also local, and \( K \)-dim \( A \times N = K \)-dim \( A \). Using Cohen’s theorem [6, (3.4)], we easily see that \( A \times N \) is Noetherian if and only if \( N \) is finitely generated.

The following proposition can be deduced from [5, (10)]; however, for the sake of completeness we include a direct proof.

(5) **Proposition.** Suppose the \( A \)-module \( N \) is nonzero. Then \( A \times N \) is self-injective if and only if \( A \) is complete and \( N \cong E(A/m) \) (i.e. the injective envelope of the residue field of \( A \)).

**Proof.** If \( A \) is complete and \( N = E(A/m) \), then by [2, p. 30], \( T = \text{Hom}_A(A \times N, N) \) is an injective \( A \times N \)-module, where \( A \times N \) is regarded as an \( A \)-module by means of the natural ring homomorphism \( A \rightarrow A \times N \). Now, by [4, (3.7)], the natural \( A \)-homomorphism \( A \rightarrow \text{Hom}_A(N, N) \) is an isomorphism. Consequently, there result \( A \)-module isomorphisms \( T \cong N \oplus \text{Hom}_A(N, N) \cong N \oplus A \), and a straightforward computation shows that the resulting isomorphism \( T \rightarrow A \times N \) is actually an \( A \times N \)-isomorphism.

Conversely, if \( A \times N \) is self-injective, then again by [2, p. 30], since \( A \times N/0 \times N \cong A \), \( \text{Hom}_A(A \times N/0 \times N, A \times N) \cong \text{Ann}_A N \times N \) is an injective \( A \)-module. Since \( A \) is local and \( N \neq 0 \), \( \text{Ann}_A N = 0 \). Furthermore the natural homomorphism \( A \rightarrow \text{Hom}_A(N, N) \) is surjective. For let \( f : N \rightarrow N \) be an \( A \)-homomorphism. The mapping \( g : 0 \times N \rightarrow A \times N \) given by \( (0, n) \mapsto (0, f(n)) \) is an \( A \times N \)-homomorphism; hence, since \( A \times N \) is self-injective, \( g \) can be extended to an \( A \times N \)-homomorphism \( g' : A \times N \rightarrow A \times N \). It follows that \( f \) is just multiplication by some element \( a \in A \). Hence, since \( \text{Ann}_A N = (0) \), \( N \) is an injective \( A \)-module for which the natural homomorphism \( A \rightarrow \text{Hom}_A(N, N) \) is an isomorphism. Using [4, (3.7)] and the now established fact that the endomorphism ring of \( N \) is local, we conclude that \( N \cong E(A/m) \) and \( A \) is complete.

(6) **Corollary.** Suppose \( A \) is an Artin local ring, \( N \neq (0) \) an \( A \)-module. \( A \times N \) is self-injective if and only if \( N \cong E(A/m) \).

We remark that a direct proof of the fact that if \( A \) is an Artin local ring, then \( A \times E(A/m) \) is self-injective appears in [3, p. 14].
Theorem. Suppose $A$ is a Cohen-Macaulay ring, having Krull dimension $n$, and $M$ a nonzero finitely generated $A$-module. Then $A \times M$ is Gorenstein if $M$ is a Gorenstein module of rank 1.

Proof. Assume first that $M$ is a Gorenstein module of rank 1. Then by [8, (3.11)], $\text{depth}_AM=\text{depth }A=n$; hence we can find $(a_1, \cdots, a_n)$ an $A$-sequence and $M$-sequence (see [8, (1.7)]). Then an easy computation shows that $(a_1, 0), \cdots, (a_n, 0)$ is an $A \times M$-sequence, and

$$A \times M/((a_1, 0), \cdots, (a_n, 0)) \cong A/(a_1, \cdots, a_n) \times M/(a_1, \cdots, a_n)M = A' \times M'.$$

Since $M$ is Gorenstein of rank 1, and $\mu_{A'}^{n+i}(m, M)=\mu_{A'}^i(m', M')$ for all $i \geq 0$ (see [1, (2.6)]), we find that $M'$ is a Gorenstein $A'$-module of rank 1. Hence $M' \cong E(A'/m')$, since $K\text{-dim }A'=0$ (see [8, (3.11)]). Now $A' \times M'$ is self-injective by (6), hence, again using [1, (2.6)], $A \times M$ is a Gorenstein ring.

Now assume conversely that $A \times M$ is Gorenstein. Let $k=\text{depth}_AM \leq n$. Let $(a_1, \cdots, a_k)$ be an $A$-sequence and $M$-sequence, and as before consider $A \times M/((a_1, 0), \cdots, (a_k, 0))=A' \times M'$. If $\text{depth }A' \times M'=n-k>0$, choose an element $(a', m')$ which is $A' \times M'$-regular. Then it is easily seen that $a'$ must be $M'$-regular, a contradiction to the fact that $\text{depth}_AM=k$. Hence $\text{depth }A' \times M'=0$, so that $n=k$; and $A' \times M'$ is self-injective. Then (6) implies that $M'=E(A'/m')$; hence $M$ is a Gorenstein module of rank 1.

Corollary. If $A$ has a Gorenstein module $M$ of rank 1, then $A$ is a quotient of a Gorenstein local ring.

Sharp has informed me that he has obtained the following extension of (2) for a commutative Noetherian ring $B$: If $B$ is Cohen-Macaulay and is a quotient of a Gorenstein ring, then $B$ has a Gorenstein module $M$ for which $\mu_{B''}(p, M)=1$ for all $p \in \text{Spec }B$. The converse of this result can be obtained from (7) by straightforward use of localization. Combining the results of these investigations, we obtain the following

Corollary. Suppose $B$ is a commutative Noetherian ring. Then there exists a Gorenstein $B$-module $M$ having the property that $\mu_{B''}(p, M)=1$ for all $p \in \text{Spec }B$ if and only if $B$ is a Cohen-Macaulay ring which can be expressed as a homomorphic image of a Gorenstein (commutative Noetherian) ring.

2 This result has been obtained independently by H. B. Foxby of Copenhagen.
REFERENCES


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