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A VECTOR MEASURE WITH NO DERIVATIVE

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ABSTRACT. Given a nonatomic scalar measure μ, there is a vector valued, μ-continuous measure of finite variation which has no derivative with respect to μ, but which has the property that the closure of its range is compact and convex.

In this note we give an example of a vector measure which answers a question raised in [5]. Further, the example can be used to gain some information about certain Banach spaces with the Radon-Nikodym property. Measure theoretic terminology is that of [5]. The Banach space (under supremum norm) of all real sequences which converge to zero is written c_0. An element of c_0 will be denoted by a doubly indexed sequence $(a_{n,i})$, with $n \geqq 1$ and $2^n \leqq i < 2^{n+1}$.

THEOREM. *Suppose that (S, Σ, μ) is a finite nonnegative measure space which has no atoms. There is a measure $\varphi: \Sigma \to c_0$ such that*

(1) *φ is μ-continuous and of finite variation,*

(2) *the closure of the range of φ is compact and convex, and*

(3) *φ has no Bochner integrable derivative with respect to μ.*

PROOF. Since μ has no atoms we may generate a doubly indexed sequence $(A_{n,i})$, $n \geqq 1$ and $2^n \leqq i < 2^{n+1}$, of measurable sets such that $\mu(A_{n,i}) = 2^{-n}\mu(S)$ and $A_{n,i}$ is the disjoint union of $A_{n+1,2i}$ and $A_{n+1,2i+1}$. Define $\varphi: \Sigma \to c_0$ by $\varphi(A) = (\mu(A \cap A_{n,i}))$. Clearly $\|\varphi(A)\| \leqq \mu(A)$ for all measurable A, so φ must be μ-continuous and of finite variation.

The range of φ is contained in the set M of sequences $(a_{n,i})$ such that $0 \leqq a_{n,i} \leqq 2^{-n}\mu(S)$ and $a_{n,i} = a_{n+1,2i} + a_{n+1,2i+1}$ for $2^n \leqq i < 2^{n+1}$. The sequences in M converge to zero uniformly so M is relatively compact, and it is clear that M is closed and convex. For $(a_{n,i}) \in M$ and $\varepsilon > 0$, choose $m \geqq 1$ so that $2^{-m}\mu(S) < \varepsilon$. Again using the fact that μ has no atoms there are measurable sets $B_i \subset A_{m,i}$, $2^m \leqq i < 2^{m+1}$, satisfying $\mu(B_i) = a_{m,i}$. Then $\|\varphi(B) - (a_{n,i})\| < \varepsilon$, where $B = \bigcup_{2^m \leqq i < 2^{m+1}} B_i$.

To see that φ has no derivative with respect to μ, we suppose the contrary. Let $f: S \to c_0$ be a derivative of φ and write $(e_{n,i})$ for the unit vectors

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in l_1. For each $i \geqq 2$ and $A \in \Sigma$,

$$\int_A \langle f(s), e_{n,i} \rangle \, \mu(ds) = \mu(A \cap A_{n,i}),$$

so there is, for each i, a μ-null set $C_i \subset S$ such that $\langle f(s), e_{n,i} \rangle = \chi_{A_{n,i}}(s)$ for $s \notin C_i$. Choose $t \in S \setminus \bigcup_i C_i$. By the way in which the sets $(A_{n,i})$ were chosen $\chi_{A_{n,i}}(t) = 1$ for infinitely many indices i, so $\lim_i \langle f(t), e_{n,i} \rangle \neq 0$. This is the desired contradiction.

A Banach space E is said to have the Radon-Nikodym property (rn) if, given any finite measure space (S, Σ, μ), every μ-continuous measure of finite variation $\varphi : \Sigma \to E$ has a Bochner integrable derivative with respect to μ (the classical examples are the reflexive spaces, separable duals and $l_1(\Gamma)$). Little is known about the structure of such Banach spaces. The example given above indicates that no Banach space E with a subspace isomorphic to c_0 has this property. Stated affirmatively **[1]** this means that each weakly unconditionally Cauchy sequence in E is unconditionally convergent. It is known **[4]** that a complemented subspace of $L_1(\mu)$ which has (rn) is isomorphic to $l_1(\Gamma)$ for some set Γ. The example given above shows that a quotient of $C(S)$ (S compact Hausdorff) with (rn) must be reflexive **[3]**, and that a complemented subspace of $C(S)$ with (rn) must be finite dimensional. Finally, let us note that this example, combined with the Radon-Nikodym theorem of Dunford and Pettis **[2]** for measures into separable duals, gives an easy proof of the classical result of Orlicz that no separable dual contains c_0.

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