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EXISTENCE AND NONUNIQUENESS OF INVARIANT MEANS ON $\mathscr{L}^\infty(\hat{G})$

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ABSTRACT. For G an infinite compact group we show the existence and nonuniqueness of invariant means on the dual of the Fourier algebra. It follows that the space of weakly almost periodic functionals on the Fourier algebra is a proper closed subspace of the dual of the Fourier algebra.

We let G denote an infinite compact group and $\hat{G}$ its dual. We use the notation of [**2**, Chapters 7 and 8]. Recall $A(G)$ denotes the Fourier algebra of G (an algebra of continuous functions on G) and $\mathscr{L}^\infty(\hat{G})$ denotes its dual under the pairing $\langle f, \phi\rangle$ ($f\in A(G)$, $\phi\in\mathscr{L}^\infty(\hat{G})$). Further note $\mathscr{L}^\infty(\hat{G})$ is identified with the C^*-algebra of bounded operators on $L^2(G)$ commuting with right translation. The module action of $A(G)$ on $\mathscr{L}^\infty(\hat{G})$ is defined by the following: for $f\in A(G)$, $\phi\in\mathscr{L}^\infty(\hat{G})$, $f\cdot\phi\in\mathscr{L}^\infty(\hat{G})$ by $\langle g, f\cdot\phi\rangle=\langle fg, \phi\rangle$, $g\in A(G)$. Also $\|f\cdot\phi\|_\infty\leqq\|f\|_A\|\phi\|_\infty$.

DEFINITION. An invariant mean on $\mathscr{L}^\infty(\hat{G})$ is a bounded linear functional p on $\mathscr{L}^\infty(\hat{G})$ such that (1) $p(\phi)\geqq 0$ whenever $\phi\geqq 0$, (2) $p(I)=1$ (I is the identity in $\mathscr{L}^\infty(\hat{G})$), and (3) $p(f\cdot\phi)=f(e)p(\phi)$, $f\in A(G)$, $\phi\in\mathscr{L}^\infty(\hat{G})$, e the identity in G. (Note $\|p\|=1$.)

If G is also abelian, then $\hat{G}$ is a discrete group and (3) is equivalent to $p(\phi_x)=p(\phi)$, $\phi\in\mathscr{L}^\infty(\hat{G})$, $x\in\hat{G}$, where ϕ_x is the translate of ϕ by x.

Let $P=\{f\in A(G): f(e)=1$ and f is positive definite$\}$. Now P is a convex spanning subset of $A(G)$ and a commutative semigroup under pointwise multiplication. Further P contains functions with supports in arbitrarily small neighborhoods of e.

The collection of invariant means on $\mathscr{L}^\infty(\hat{G})$ will be denoted by $\mathscr{N}$. Thus for $p\in\mathscr{L}^\infty(\hat{G})^*$ (the dual of $\mathscr{L}^\infty(\hat{G})$), $p\in\mathscr{N}$ if and only if (1) $p(\phi)\geqq 0$ whenever $\phi\geqq 0$, (2) $p(I)=1$, and (3) $p(f\cdot\phi)=p(\phi)$, $f\in P$, $\phi\in\mathscr{L}^\infty(\hat{G})$.

Note $\langle f, I\rangle=f(e)$, and $f\cdot I=f(e)I$ ($f\in A(G)$).

In $\mathscr{L}^\infty(\hat{G})$ let w^* denote the weak-* topology $\sigma(\mathscr{L}^\infty(\hat{G}), A(G))$. In $\mathscr{L}^\infty(\hat{G})^*$ let τ denote the weak-* topology $\sigma(\mathscr{L}^\infty(\hat{G})^*, \mathscr{L}^\infty(\hat{G}))$. Let j denote the canonical embedding of $A(G)$ into $\mathscr{L}^\infty(\hat{G})^*$.

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THEOREM 1. *$\mathscr{N}$ is not empty.*

PROOF. Let $K=\{q\in\mathscr{L}^\infty(\hat{G})^*:q(\phi)\geqq 0$ whenever $\phi\geqq 0$ $(\phi\in\mathscr{L}^\infty(\hat{G}))$ and $q(I)=1\}$. Then K is a τ-compact convex subset of $\mathscr{L}^\infty(\hat{G})^*$. Further P acts as a commutative semigroup of τ-continuous operators of K into K; the action is $(f,q)\mapsto f\cdot q$ where $(f\cdot q)(\phi)=q(f\cdot\phi)$, for $f\in P$, $q\in K$, and $\phi\in\mathscr{L}^\infty(\hat{G})$. By the Markov-Kakutani fixed point theorem [**1**, p. 456] there exists $p\in K$ such that $f\cdot p=p$ for all $p\in K$. But then $p(f\cdot\phi)=p(\phi)$ for all $\phi\in\mathscr{L}^\infty(\hat{G})$, so $p\in\mathscr{N}$. □

For $\phi\in\mathscr{L}^\infty(\hat{G})$, let $O^*(\phi)$ be the w^*-closure in $\mathscr{L}^\infty(\hat{G})$ of $\{f\cdot\phi : f\in P\}$. We will characterize the fixed points on $O^*(\phi)$, that is, the constant multiples of I which are in $O^*(\phi)$. Observe that on $O^*(\phi)$ the w^* topology is the topology of pointwise convergence on $\hat{G}$. (See [**4**] for a more general setting.)

THEOREM 2. *Let $\phi\in\mathscr{L}^\infty(\hat{G})$ and $\{g_\alpha\}$ be a net in P such that $g_\alpha\cdot\phi\xrightarrow{\alpha} cI$ in w^*, $c\in\mathbf{C}$. Then there exists $p\in\mathscr{N}$ with $p(\phi)=c$.*

PROOF. Let $g_\alpha\cdot\phi\xrightarrow{\alpha} cI$ in w^*. For a neighborhood V of e, let $u_V\in P$ with $\operatorname{spt} u_V\subset V$. For λ a w^*-neighborhood of cI, write $g_{\lambda,V}=g_\alpha u_V$ where $u_V g_\alpha\cdot\phi\in\lambda$. (Note $g_\alpha u_V\cdot\phi\xrightarrow{\alpha} cI$ in w^* for each V.) Thus

$$\operatorname{spt} g_{\lambda,V}\xrightarrow{(\lambda,V)}\{e\}\quad\text{and}\quad g_{\lambda,V}\cdot\phi\xrightarrow{(\lambda,V)} cI$$

in w^*. Thus we may assume that $\operatorname{spt} g_\alpha\xrightarrow{\alpha}\{e\}$.

Let $p\in\mathscr{L}^\infty(\hat{G})^*$ be a τ-cluster point of $\{jg_\alpha\}$. Since $\operatorname{spt} g_\alpha\xrightarrow{\alpha}\{e\}$, $p\in\mathscr{N}$. Let $\varepsilon>0$. Pick α_0 such that $\alpha\geqq\alpha_0$ implies

$$|\langle g_\alpha,\phi\rangle - c| = |\langle 1, g_\alpha\cdot\phi\rangle - \langle 1, cI\rangle| < \varepsilon/2.$$

Choose $\alpha\geqq\alpha_0$ with $|\langle g_\alpha,\phi\rangle - p(\phi)|<\varepsilon/2$. Then

$$|p(\phi)-c| \leqq |p(\phi)-\langle g_\alpha,\phi\rangle| + |\langle g_\alpha,\phi\rangle - c| < \varepsilon$$

and so $p(\phi)=c$. □

COROLLARY 3. *If there is some $\phi\in\mathscr{L}^\infty(\hat{G})$ such that $\{c\in\mathbf{C}: cI\in O^*(\phi)\}$ has at least two points, then there are at least two different invariant means on $\mathscr{L}^\infty(\hat{G})$, hence infinitely many.*

PROOF. $\mathscr{N}$ is a convex subset of $\mathscr{L}^\infty(\hat{G})^*$. □

THEOREM 4. *Given $p\in\mathscr{N}$, there is a net $\{g_\alpha\}\subset P$ such that $g_\alpha\cdot\phi\xrightarrow{\alpha} p(\phi)I$ in w^* for all $\phi\in\mathscr{L}^\infty(\hat{G})$.*

PROOF. Let $p\in\mathscr{N}$. Now p as a linear functional on $\mathscr{L}^\infty(\hat{G})$ has norm one, thus by Goldstine's theorem [**1**, p. 424], there is a net $\{h_\alpha\}$ in the unit ball of $A(G)$ such that $jh_\alpha\xrightarrow{\alpha} p$ in τ.

Write $h_\alpha = h_{\alpha_1} - h_{\alpha_2} + ih_{\alpha_3} - ih_{\alpha_4}$, h_{α_n} positive definite, $1 \leqq n \leqq 4$. For $\phi \in \mathscr{L}^\infty(\hat{G})$ with $\phi = \phi^*$, the numbers $p(\phi)$, $\langle h_{\alpha_1} - h_{\alpha_2}, \phi \rangle$, and $\langle h_{\alpha_3} - h_{\alpha_4}, \phi \rangle$ are all real. Thus $\langle h_{\alpha_3} - h_{\alpha_4}, \phi \rangle \xrightarrow{\alpha} 0$. Thus we may assume $h_\alpha = h_{\alpha_1} - h_{\alpha_2}$ with $h_{\alpha_1}(e) + h_{\alpha_2}(e) \leqq 1$. Since $jh_\alpha \xrightarrow{\alpha} p$ in τ, $h_{\alpha_1}(e) - h_{\alpha_2}(e) \xrightarrow{\alpha} 1$ (put $\phi = I$), hence $h_{\alpha_1}(e) \xrightarrow{\alpha} 1$ and $h_{\alpha_2}(e) \xrightarrow{\alpha} 0$. Let $g_\alpha = h_{\alpha_1}/h_{\alpha_1}(e)$ (for α sufficiently large), then $g_\alpha \in P$. And for $f \in P$, $\phi \in \mathscr{L}^\infty(\hat{G})$, $\langle f, g_\alpha \cdot \phi \rangle = \langle g_\alpha, f \cdot \phi \rangle \xrightarrow{\alpha} p(f \cdot \phi) = p(\phi)$. Thus $g_\alpha \cdot \phi \xrightarrow{\alpha} p(\phi) I$ in w^*. □

Corollary 5. *For $\phi \in \mathscr{L}^\infty(\hat{G})$, $\{p(\phi) : p \in \mathscr{N}\} = \{c \in \mathbf{C} : cI \in O^*(\phi)\}$.*

Theorem 6. *Let H be a closed subgroup of G and suppose p is an invariant mean on $\mathscr{L}^\infty(\hat{H})$. Then p extends to $\mathscr{L}^\infty(\hat{G})$, that is, there is an invariant mean q on $\mathscr{L}^\infty(\hat{G})$ with $q | \rho^* \mathscr{L}^\infty(\hat{H}) = p$ where ρ^* is the adjoint of $\rho : A(G) \to A(H)$, the restriction map, which is bounded and onto (see* [**2**], Chapter 8]).

Proof. Previously defined symbols subscripted with H denote the appropriate objects with respect to H. By Theorem 4, there exists a net $\{f_\alpha\} \subset P_H$ such that $f_\alpha \cdot \phi \xrightarrow{\alpha} p(\phi) I_H$ in w_H^* for all $\phi \in \mathscr{L}^\infty(\hat{H})$. Let V be a neighborhood of e in G and let $u_V \in P$ with spt $u_V \subset V$. For λ a τ_H-neighborhood of p, let $f_{\lambda,V} = f_\alpha u_V | H$ where $j_H(f_\alpha u_V | H) \in \lambda$. (Note that $\langle f_\alpha u_V | H, \phi \rangle = \langle u_V | H, f_\alpha \cdot \phi \rangle \xrightarrow{\alpha} \langle u_V | H, p(\phi) I_H \rangle = p(\phi)$, $\phi \in \mathscr{L}^\infty(\hat{H})$ and V fixed.) Now let $g_\alpha \in P$ with $\rho g_\alpha = g_\alpha | H = f_\alpha$. Define $g_{\lambda,V} = g_\alpha u_V$. Choose q a τ-cluster point of $\{ jg_{\lambda,V} \}$. Then $q \in \mathscr{N}$ by Theorem 2.

For $\phi \in \mathscr{L}^\infty(\hat{H})$ and $\varepsilon > 0$, pick λ_0 such that for $\lambda \geqq \lambda_0$ (that is, $\lambda \subset \lambda_0$) and any V, $|\langle f_{\lambda,V}, \phi \rangle - p(\phi)| < \varepsilon/2$. Now choose $(\lambda, V) \geqq (\lambda_0, V)$ with $|q(\rho^* \phi) - \langle g_{\lambda,V}, \rho^* \phi \rangle| < \varepsilon/2$. Thus

$$\begin{aligned} |q(\rho^*\phi) - p(\phi)| &\leqq |q(\rho^*\phi) - \langle g_{\lambda,V}, \rho^*\phi \rangle| + |\langle g_{\lambda,V}, \rho^*\phi \rangle - p(\phi)| \\ &< \varepsilon/2 + |\langle f_{\lambda,V}, \phi \rangle - p(\phi)| < \varepsilon. \end{aligned}$$

Hence $q(\rho^*\phi) = p(\phi)$ for $\phi \in \mathscr{L}^\infty(\hat{H})$. □

Corollary 7. *Let H be a closed subgroup of G with two different invariant means on $\mathscr{L}^\infty(\hat{H})$. Then there are two different invariant means on $\mathscr{L}^\infty(\hat{G})$.*

Corollary 8. *Let G be a compact Lie group. Then there are two different invariant means on $\mathscr{L}^\infty(\hat{G})$.*

Proof. Compact Lie groups have a torus T as a closed abelian subgroup. And invariant means are not unique on T, for 0 and 1 are in $O^*(\phi_{Z^+})$ (ϕ_{Z^+} is the characteristic function of the nonnegative integers). Now use Corollary 3. □

REMARK. If $G=SU(2)$, the group of 2×2 unitary matrices of determinant one, then we can construct $\phi\in\mathscr{L}^\infty(\hat{G})$ with $O^*(\phi)$ containing two different constants. The dual of $SU(2)$ can be identified with the set Z_+, and for $k=1, 2, \cdots$, choose integers n_k, m_k such that the subsets E_k, $F_k\subset Z_+$, $E_k=\{n_k-k, \cdots, n_k-1, n_k, n_k+1, \cdots, n_k+k\}$, $F_k=\{m_k-k, \cdots, m_k-1, m_k, m_k+1, \cdots, m_k+k\}$, are pairwise disjoint and cover Z_+. Let $\phi\in\mathscr{L}^\infty(\hat{G})$ be such that $\phi_n=I_{n+1}$ (identity operator on $\mathbf{C}^{n+1}$) for $n\in\bigcup E_k$ and $\phi_n=0$ for $n\in\bigcup F_k$. Then $O^*(\phi)$ contains 0 and 1.

For α, $\beta\in\hat{G}$, the tensor product, $T_\alpha\otimes T_\beta$, of the two representations decomposes into irreducible components: $T_\alpha\otimes T_\beta\cong\sum\oplus_\gamma M_{\alpha\beta}(\gamma)T_\gamma$, where $M_{\alpha\beta}(\gamma)=\int_G \chi_\alpha\chi_\beta\bar{\chi}_\gamma\, dm_G$, a nonnegative integer (χ_α is the character of the class α and m_G is normalized Haar measure on G). For E, $F\subset\hat{G}$, we define $E\otimes F=\{\gamma\in\hat{G}: M_{\alpha\beta}(\gamma)\neq 0,\ \alpha\in E,\ \beta\in F\}$. This operation makes $\hat{G}$ into a hypergroup. If $E\otimes E\subset E$, then E is called a subhypergroup of $\hat{G}$. For $\alpha\in\hat{G}$, there is a conjugate $\bar{\alpha}\in\hat{G}$ such that $\chi_{\bar{\alpha}}(x)=\bar{\chi}_\alpha(x)$, $x\in G$. If E is a subhypergroup and $\bar{E}=\{\bar{\alpha}:\alpha\in E\}\subset E$, then E is called a normal subhypergroup.

For $F\subset\hat{G}$, $[F]$ denotes the smallest normal subhypergroup containing F. We say that $\hat{G}$ is finitely generated if and only if there is a finite set $\alpha_1, \cdots, \alpha_k\in\hat{G}$ such that $[\alpha_1, \cdots, \alpha_k]=\hat{G}$.

Let $\phi\in\mathscr{L}^\infty(\hat{G})$, define the carrier of ϕ, cr ϕ, to be the set $\{\alpha\in\hat{G}:\phi_\alpha\neq 0\}$. For $f\in A(G)$, and $\phi\in\mathscr{L}^\infty(\hat{G})$, $\mathrm{cr}(f\cdot\phi)=(\mathrm{cr}\, f)^-\otimes(\mathrm{cr}\,\phi)$ (see [3]).

THEOREM 9. *Let G be an infinite compact group. Then there are nonunique invariant means on* $\mathscr{L}^\infty(\hat{G})$.

PROOF. If G is a Lie group, then $\mathscr{L}^\infty(\hat{G})$ has nonunique invariant means by Corollary 8. So we assume $\hat{G}$ is not finitely generated.

Let $\nu=\mathrm{card}\,\hat{G}$ (the cardinality of $\hat{G}$). Then ν is an infinite limit ordinal, and let $\hat{G}=\{\gamma_0, \gamma_1, \cdots, \gamma_\lambda, \cdots\}$ $(\lambda<\nu)$ be a well-ordering of $\hat{G}$. Let $\alpha_0=\gamma_0$. For $\lambda<\nu$, let $\alpha_\lambda\in\hat{G}$ be the least element of $\hat{G}$ relative to the well-ordering with $\alpha_\lambda\notin\bigcup\{[\alpha_0, \cdots, \alpha_\mu]:\mu<\lambda\}$. (The existence is assured since for $\lambda<\nu$ with λ infinite, card $\bigcup\{[\alpha_0, \cdots, \alpha_\mu]:\mu<\lambda\}=\mathrm{card}\,\lambda\leqq\lambda<\nu$.)

Define an ordinal-valued function i on $\hat{G}$ by $i(\alpha)=\inf\{\lambda:\alpha\in[\alpha_0, \cdots, \alpha_\lambda]\}$ $(\alpha\in\hat{G})$. If α, $\beta\in\hat{G}$ with $i(\alpha)>i(\beta)$, then $i(\gamma)=i(\alpha)$ for all $\gamma\in\alpha\otimes\beta$ (recall $M_{\alpha\beta}(\gamma)=M_{\gamma\beta}(\alpha)$, α, β, $\gamma\in\hat{G}$).

Call a limit ordinal even, and a nonlimit ordinal even (respectively odd) if its predecessor is odd (respectively even). Let $E=\{\alpha\in\hat{G}: i(\alpha) \text{ is even}\}$.

Let $\phi\in\mathscr{L}^\infty(\hat{G})$ be defined by $\phi_\alpha=I_{n_\alpha}$ (the identity operator in $\mathscr{B}(C^{n_\alpha})$) for $\alpha\in E$ and $\phi_\alpha=0$ otherwise. We first show $0\in O^*(\phi)$. For each α_λ, let $f_\lambda=\chi_{\alpha_\lambda}/n_{\alpha_\lambda}$. Then $f_\lambda\in P$. If $\alpha\in\mathrm{cr}(f_\lambda\cdot\phi)$, then $(\alpha_\lambda\otimes\alpha)\cap E\neq\varnothing$. Now consider $\{\alpha_\lambda:\lambda<\nu, \lambda \text{ odd}\}$ as a net. Fix $\alpha\in\hat{G}$, then for all odd $\lambda<\nu$ with

$\lambda > i(\alpha)$, $i(\gamma)$ is odd for each $\gamma \in \alpha \otimes \alpha_\lambda$; and so $\alpha \otimes \alpha_\lambda \cap E = \varnothing$. Thus $(f_\lambda \cdot \phi)_\alpha = 0$ for all large odd λ, and so $(f_\lambda \cdot \phi) \xrightarrow{\lambda} 0$ pointwise on $\hat{G}$. Thus $f_\lambda \cdot \phi \xrightarrow{\lambda} 0$ in w^*, and so $0 \in O^*(\phi)$.

Now let $\psi = I - \phi$. A similar argument yields $0 \in O^*(\psi)$. Thus there are invariant means p and q on $\mathscr{L}^\infty(\hat{G})$ with $p(\phi) = 0$ and $q(\psi) = 0$. But $q(\phi) = q(I - \psi) = 1 - 0 = 1$. Thus there are nonunique means on $\mathscr{L}^\infty(\hat{G})$. $\square$

Definition. Let $\phi \in \mathscr{L}^\infty(\hat{G})$. We call ϕ a weakly almost periodic functional if and only if the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ is a weakly compact operator. The space of all such is denoted $W(\hat{G})$. If the map $f \mapsto f \cdot \phi$ is a compact operator, we call ϕ an almost periodic functional. The space of all such is denoted $AP(\hat{G})$.

Theorem 10. *Let G be an infinite compact group. Then $W(\hat{G})$ is a proper closed subspace of $\mathscr{L}^\infty(\hat{G})$.*

Proof. In [3] we showed that $W(\hat{G})$ is a closed subspace of $\mathscr{L}^\infty(\hat{G})$ with a unique invariant mean (by using Eberlein's ergodic theory). $\square$

Remark. The group $G = SU(2)$ provides an example of a compact group such that there are nonzero continuous measures whose Fourier-Stieltjes transforms are in $AP(\hat{G})$, a remarkable contrast to the abelian case. We can in fact show $\mathscr{C}_0(\hat{G}) \subset AP(\hat{G})$, where $\mathscr{C}_0(\hat{G}) = \{\phi \in \mathscr{L}^\infty(\hat{G})$: for $\varepsilon > 0$, there are only finitely many $\alpha \in \hat{G}$ with $\|\phi_\alpha\|_\infty \geqq \varepsilon\}$.

Since $AP(\hat{G})$ is an $A(G)$-module it suffices to show that the Fourier-Stieltjes transform $\hat{m}_G$ of the Haar measure on G is in $AP(\hat{G})$. For $k \in Z_+$, let P_k denote the map from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ by $(P_k f)_n = (\check{f})^\wedge_n$ ($\check{f}(x) = f(x^{-1})$, $x \in G$) for $0 \leqq n \leqq k$, and $(P_k f)_n = 0$ for $n > k$. Now P_k is of finite rank, and so it suffices to show that

$$\|P_k(f) - f \cdot \hat{m}_G\|_\infty < \frac{1}{k+1} \|f\|_A \qquad (f \in A(G)).$$

But

$$\begin{aligned}\|P_k(f) - f \cdot \hat{m}_G\|_\infty &\leqq \max\{\|(\check{f})^\wedge_n\|_\infty : n > k\} \\ &\leqq \max\{\|(\check{f})^\wedge_n\|_1 : n > k\} \leqq \frac{1}{k+2} \|f\|_A,\end{aligned}$$

since $\|f\|_A = \|\check{f}\|_A = \sum_{n=0}^\infty (n+1) \|(\check{f})^\wedge_n\|_1$.

The same method proves the following:

Theorem 11. *If G is a compact group such that for each $l = 1, 2, \cdots$, there are only finitely many $\alpha \in \hat{G}$ with $l = n_\alpha$ (the degree of α), then $\mathscr{C}_0(\hat{G}) \subset AP(\hat{G})$.*

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