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PEANO CURVES IN FUNCTION ALGEBRAS

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ABSTRACT. We give a short proof of the following result which was obtained by Pełczyński. If X is an uncountable, compact metric space and if A is a function algebra on X, then there exists f in A such that $f(X)$ has interior in the plane.

Let X be a compact metric space and let A be a function algebra on X. If one restricts the "size" of $f(X)$ for each f in A, then the space X is likewise restricted. For example, Rudin [4] has shown that $f(X)$ is countable for each f in A if and only if X is countable. Of course, $f(X)$ is countable for each f in A implies $A=C(X)$. Pełczyński [2] has shown that X is uncountable if and only if A contains a closed subspace M and X contains a perfect, closed subset K such that $f\to f|K$ is an isometry of M onto $C(K)$. Our purpose is to give a different approach to some of Pełczyński's work.

If X is a compact subset of the plane $\mathbf{C}$, then $P(X)$ denotes the uniform closure in $C(X)$ of the polynomials. We begin with the following result.

THEOREM. *Suppose X is an uncountable, compact subset of $\mathbf{C}$. There is a closed, uncountable subset K of X such that K is a peak set for $P(X)$ and $P(X)|K=C(K)$.*

PROOF. We may assume that X is polynomially convex. By Wermer's characterization [6] of the annihilator of $P(X)$, there is a nonnegative measure μ on ∂X, the boundary of X, such that $\nu\perp P(X)$ and ν is supported on ∂X implies that ν is absolutely continuous with respect to μ. Hence, by Bishop's generalization [1] of the Rudin-Carleson theorem, we only need to find $K\subseteq\partial X$ such that $\mu(K)=0$ and K is closed and uncountable. Such a K exists. Namely, since ∂X is uncountable, there is a continuous map g on ∂X onto $[0, 1]\times[0, 1]$. Then $\mu(g^{-1}(\{x\}\times[0, 1]))=0$ for all but countably many x in $[0, 1]$.

COROLLARY. *Suppose Y is a compact metric space and A is a function algebra on Y. If $f(Y)$ has no interior in $\mathbf{C}$ for each f in A, then Y is countable and hence $A=C(Y)$.*

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PROOF. Suppose $f(Y)=X$ is uncountable for some f in A. Then by the above result there is $g \in P(X)$ such that $g(X)$ has interior in C. Hence, $(g \circ f)(Y)=g(X)$ has interior in C and $g \circ f$ belongs to A which is a contradiction. Therefore, $f(X)$ is countable for each f in A and Rudin's result [4] implies Y is countable.

Note. Much more can be said in the case of the disc algebra, the algebra of continuous functions on the unit circle T which have analytic extensions to the open unit disc. For example ([3], [5]), there is a function of the form $f(z)=\sum_{n=1}^{\infty} a_n z^n$ where $\sum_{n=1}^{\infty} a_n z^n$ converges absolutely on T and $f(T)$ has interior in the plane.

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