

ON CHARACTERIZATION OF RIEMANNIAN MANIFOLDS BY GROWTH OF TUBULAR NEIGHBORHOODS¹

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ABSTRACT. If the area function of the tubular neighborhoods of a compact submanifold of a Riemannian manifold satisfies a certain linear differential inequality, then the codimension of the submanifold is at most the order of that inequality.

1. Let N be a Riemannian manifold of dimension $n \geq 2$ and let M be a compact orientable submanifold of dimension m embedded in N . (All manifolds, maps, etc. are supposed smooth.) For $s > 0$, let M_s denote the set of points lying on geodesics normal to M and at arc length s from M . For sufficiently small s , M_s is a smooth hypersurface in N . We denote by $\mathcal{A}(s)$ the area of M_s . H. Wu [2] derived an elegant formula for $\mathcal{A}''(s)$ and used it to characterize the Euclidean plane amongst Riemannian manifolds. Later he and R. A. Holzinger [1] proved the following more encompassing characterization:

A Riemannian manifold has the property that the growth function \mathcal{A} of each one of its compact hypersurfaces satisfies the linear differential equation

$$\mathcal{A}''(s) + c\mathcal{A}(s) = 0$$

(where c is a fixed constant) if and only if it is a two-dimensional Riemannian manifold of constant curvature equal to c .

In this note we obtain a formula for $\mathcal{A}''(s)$ and indeed for all the derivatives of \mathcal{A} , valid for submanifolds M of any dimension. Our method yields \mathcal{A}'' in a simpler, more mechanical fashion than does Wu's. We can easily reprove the theorem of Holzinger and Wu. We also obtain an extension of their theorem to the case when \mathcal{A} satisfies a linear differential inequality of higher order with "nice" coefficients. We show the order of this differential inequality is an upper bound for the codimensions of the submanifolds involved.

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2. Let M be a fixed compact orientable submanifold of N with an embedded tubular neighborhood U in N . Let $U^* = U - N$. Our arguments are local, so we may suppose N to be oriented. Choose a positively oriented (local) orthonormal frame field e_1, \dots, e_n on U^* , supposed *adapted* so e_1 is the tangent vector of geodesics leaving M normally, while each e_i ($2 \leq i \leq n$) is parallel along such a geodesic. Let $\omega_1, \dots, \omega_n$ be the dual co-frame and let ω_{ij} ($1 \leq i, j \leq n$) be the connection 1-forms of the Riemannian structure of N as restricted to U^* . The forms ω_i, ω_{ij} satisfy the Cartan structural equations

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j$$

and

$$d\omega_{ij} = \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},$$

where Ω_{ij} is the curvature 2-form.

For sufficiently small $s > 0$, $M_s \subset U^*$ and is a hypersurface with normal field e_1 so there are functions $h_{ij} = h_{ji}$ ($2 \leq i, j \leq n$) on U^* with

$$\omega_{1i} = \sum_{j=2}^n h_{ij} \omega_j.$$

The *mean curvature* is given at $p \in M_s$ by

$$H(p) = \frac{1}{n-1} \sum_{j=2}^n h_{jj}(p).$$

(Note that the signs have been chosen so that M_1 has $H=1$ when M is the origin in the Euclidean plane.) Finally, the function $\mathcal{A}(s)$ is given by

$$\mathcal{A}(s) = \int_{M_s} \omega_2 \wedge \dots \wedge \omega_n.$$

We use Stokes' theorem on the closed set $M_{s+\delta, s}$ included between M_s and $M_{s+\delta}$ ($\delta > 0$). We then have

$$\mathcal{A}(s + \delta) - \mathcal{A}(s) = \int_{M_{s+\delta, s}} d(\omega_2 \wedge \dots \wedge \omega_n).$$

Using the first Cartan structural equation, we calculate

$$d(\omega_2 \wedge \dots \wedge \omega_n) = (n-1)H\omega_1 \wedge \dots \wedge \omega_n.$$

Hence

$$\mathcal{A}(s + \delta) - \mathcal{A}(s) = \int_{M_{s+\delta, s}} (n-1)H\omega_1 \wedge \dots \wedge \omega_n.$$

Keeping in mind the geometrical meaning of ω_1 , we divide by δ and let $\delta \rightarrow 0$ to obtain

$$\mathcal{A}'(s) = \int_{M_s} (n - 1)H\omega_2 \wedge \cdots \wedge \omega_n.$$

In order to calculate higher derivatives of \mathcal{A} , it will be convenient to use this notation: if F is a smooth function on U^* , then

$$dF = \sum_{j=1}^n F_j \omega_j.$$

Again applying Stokes' theorem, we have

$$\mathcal{A}'(s + \delta) - \mathcal{A}'(s) = \int_{M_{s+\delta, s}} (n - 1) d(H\omega_2 \wedge \cdots \wedge \omega_n).$$

It follows easily that

$$\mathcal{A}''(s) = \int_{M_s} (n - 1)\{H_1 + (n - 1)H^2\}\omega_2 \wedge \cdots \wedge \omega_n.$$

There is a general formula:

$$\mathcal{A}^{(k)}(s) = \int_{M_s} I_k \omega_2 \wedge \cdots \wedge \omega_n.$$

The functions I_k satisfy the recurrence relation

$$I_{k+1} = I_{k,1} + (n - 1)HI_k.$$

We will not attempt to give an explicit formula for I_k in terms of H and its derivatives.

3. Now we will extend the theorem of Holzsager and Wu. The following estimate generalizes the lemma of Wu [2].

LEMMA 1. *As $s \rightarrow 0$, $H \sim (n - m - 1)/(n - 1) \cdot 1/s$. This asymptotic relation is uniform from one normal geodesic to another and may be differentiated as often as desired.*

(Here we use the asymptotic symbol \sim in the usual sense: $f \sim g$ means $f(s)/g(s) \rightarrow 1$ as $s \rightarrow 0$.)

Verification of the asymptotic relation of Lemma 1 along a given normal geodesic proceeds from an elementary calculation using the first and second Cartan structural equations. We obtain relations expressing the successive derivatives of each h_{ii} in terms of the lower-order radial derivatives of h_{ii} as well as the Riemannian curvature and its derivatives. Using the smoothness of the data, we establish inductively that each radial derivative of h_{ii} has the proper order of infinity as $s \rightarrow 0$. Knowing this, we can use the

uniqueness of asymptotic expansion to calculate the asymptotic behavior of the derivatives of H as $s \rightarrow 0$. The compactness of M allows the conclusion that these asymptotic estimates are uniform over M from one normal geodesic to another. The perturbation techniques used are by now standard in global analysis.

Using the recurrence relation for I_k and Lemma 1, we can prove an estimate for I_k by induction.

LEMMA 2. As $s \rightarrow 0$,

$$I_k \sim (n - m - 1) \cdots (n - m - k)s^{-k},$$

and this estimate is uniform from one normal geodesic to another.

Here is our main result.

THEOREM. Let k be a positive integer and fix $m \leq n - 2$. Suppose for each compact M of dimension m there are bounded continuous functions c_1^M, \dots, c_k^M such that the area function \mathcal{A} of M satisfies the differential inequality

$$\mathcal{A}^{(k)}(s) + \sum_{j=1}^k c_j^M(s) \mathcal{A}^{(k-j)}(s) \leq 0$$

for sufficiently small $s > 0$. Then $n - m \leq k$.

The proof is based upon Lemma 2. We have

$$\mathcal{A}^{(j)}(s) = \int_{M_s} I_j \omega_2 \wedge \cdots \wedge \omega_n$$

whence the differential inequality hypothesized implies

$$\int_{M_s} \left\{ I_k + \sum_{j=1}^k c_j^M(s) I_{k-j} \right\} \omega_2 \wedge \cdots \wedge \omega_n \leq 0.$$

By Lemma 2, the function in curly brackets has the uniform asymptotic expression

$$\{\cdots\} = (n - m - 1) \cdots (n - m - k)s^{-k} + o(s^{-k})$$

as $s \rightarrow 0$. If $n - m > k$, $\{\cdots\} > 0$ for sufficiently small $s > 0$, so

$$\int_{M_s} \{\cdots\} \omega_2 \wedge \cdots \wedge \omega_n > 0.$$

This contradiction shows $n - m \leq k$, as was claimed.

4. We show how to obtain the theorem of Holzsager and Wu. Suppose

$$\mathcal{A}''(s) + c\mathcal{A}(s) = 0$$

for some constant c . We apply our theorem with submanifolds M consisting of single points, so $m=0$. We conclude $n=2$. Calculating I_2 explicitly, we find

$$\mathcal{A}''(s) + c\mathcal{A}(s) = \int_{M_s} (H_1 + H^2 + c)\omega_2.$$

In our case, M_s is a curve. In terms of the adapted coframe field, $\omega_{12} = H\omega_2$. The Gauss curvature function K satisfies

$$d\omega_{12} = -K\omega_1 \wedge \omega_2$$

from which we obtain $H_1 + H^2 = -K$. Therefore,

$$\int_{M_s} (c - K)\omega_2 = 0.$$

Since M and s are arbitrary, $K \equiv c$ on N .

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