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A NOTE ON THE UNIQUENESS OF RINGS OF COEFFICIENTS IN POLYNOMIAL RINGS

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ABSTRACT. We say that the ring A is of transcendence degree n over its subfield k if for every prime $P \subset A$ the transcendence degree of A/P over k is at most n and equality is attained for some P. In this paper we prove the following: Suppose A is a noetherian ring of transcendence degree one over its subfield k. Then if B is any ring such that the polynomial rings

$$A[X_1, \cdots, X_m] \quad \text{and} \quad B[Y_1, \cdots, Y_m]$$

are isomorphic, A is isomorphic to B. Moreover if A has no non-trivial idempotents then either A is isomorphic to the polynomials in one variable over a local artinian ring or, modulo the nil radical, the given isomorphism takes A onto B.

This is a continuation of the investigation, begun in **[CE]** and **[AEH]**, of the extent to which a ring is determined by its polynomial rings. More precisely, we study the following: Suppose A is a ring and $R=A[X_1, \cdots, X_n]$ is a ring of polynomials over A. If B is a ring and $S=B[Y_1, \cdots, Y_n]$ is a ring of polynomials over B such that R is isomorphic to S, does it follow that A is isomorphic to B? In **[CE]** and **[AEH]** it is shown that for a fairly extensive class of rings A, the answer is affirmative. Our main purpose here is to extend this class of rings to include rings of Krull dimension one which are finitely generated over a field. This has been done for the case of rings without zero divisors in **[AEH]**.

Unless there is a statement to the contrary all rings are assumed to be commutative and to possess an identity $1\neq 0$. If R is a ring, we write $A[X_1, \cdots, X_n]=A^{(n)}$ when we want it to be understood or emphasized that the X_i's are algebraically independent over A. If A is a ring, we let $N(A)$ denote the nil radical of A and A^* the reduced ring $A/N(A)$.

Let A be a ring and $R=A[X_1, \cdots, X_n]=A^{(n)}$. We have

$$N(R) = N(A)(R[X_1, \cdots, X_n]) \quad \text{and} \quad R^* = A^*[\bar{X}_1, \cdots, \bar{X}_n] = A^{*(n)}.$$

If A and B are rings and there is an isomorphism

$$A^{(n)} = A[X_1, \cdots, X_n] \xrightarrow{\sigma} B[Y_1, \cdots, Y_n] = B^{(n)},$$

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then since σ maps the nil radical of $A^{(n)}$ onto the nil radical of $B^{(n)}$ we have an induced isomorphism σ^* and a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N(A)[X_1, \cdots, X_n] & \xrightarrow{\sigma} & N(B)[Y_1, \cdots, Y_n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A[X_1, \cdots, X_n] & \xrightarrow{\sigma} & B[Y_1, \cdots, Y_n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^*[\bar{X}_1, \cdots, \bar{X}_n] & \xrightarrow{\sigma^*} & B^*[\bar{Y}_1, \cdots, \bar{Y}_n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

With this notation we give the following:

(1) DEFINITION. A ring A is said to be *strongly invariant* if for any ring B and any isomorphism of polynomial rings $\sigma: A[X_1, \cdots, X_n] \to B[Y_1, \cdots, Y_n]$ we have $\sigma(A^*) = B^*$.

A weaker concept is:

(2) DEFINITION. A ring A is said to be *invariant* if for any ring B such that there is an isomorphism of the polynomial rings $A[X_1, \cdots, X_n]$ and $B[Y_1, \cdots, Y_n]$, we have B isomorphic to A.

Definition (1) is consistent with those of **[CE]** and **[AEH]**. In the case where A has no nilpotent elements, and in particular where A is a domain, (1) simply asserts that the isomorphism σ takes A onto B. With these definitions it is not obvious that strongly invariant rings are invariant. Before proving this we make the following useful observation.

(3) If A is a ring, then A is invariant (resp. strongly invariant) if and only if whenever B is a ring such that $R = A^{(n)} = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n] = B^{(n)}$ then A is isomorphic to B (resp. $A^* = B^*$ in R^*).

The proof is straightforward and we omit it.

(4) Strongly invariant rings are invariant.

PROOF. Suppose A is a strongly invariant ring. Let $R = A[X_1, \cdots, X_n] = A^{(n)} = B[Y_1, \cdots, Y_n] = B^{(n)}$ for some ring B, then $A^* = B^*$ by (3). We claim this implies that $A[X_1, \cdots, X_n] = A[Y_1, \cdots, Y_n]$. Since $A^{*(n)} = A^*[\bar{X}_1, \cdots, \bar{X}_n] = A^*[\bar{Y}_1, \cdots, \bar{Y}_n]$, we have, for each i, $\bar{X}_i = \bar{g}_i(\bar{Y}_1, \cdots, \bar{Y}_n)$, so

$$X_i = g_i(Y_1, \cdots, Y_n) + n_i(X_1, \cdots, X_n)$$

where n_i has nilpotent coefficients which are elements of A. We also have, for each i,

$$Y_i = h_i(X_1, \cdots, X_n).$$

Let Δ be the set of all the coefficients of the g_i, n_j and h_k. Set $A_0=\Pi[\Delta]$ where Π is the prime subring of A. Then

$$A_0[Y_1, \cdots, Y_n] \subseteq A_0[X_1, \cdots, X_n] = A_0^{(n)}$$

and

$$A_0^*[\bar{Y}_1, \cdots, \bar{Y}_n] = A_0^*[\bar{X}_1, \cdots, \bar{X}_n].$$

Now $A_0[Y_1, \cdots, Y_n]=A_0[X_1, \cdots, X_n]$. For

$$A_0[X_1, \cdots, X_n] \subseteq A_0[Y_1, \cdots, Y_n] + N(A_0)[Y_1, \cdots, Y_n],$$

and iterating this we have

$$A_0[X_1, \cdots, X_n] \subseteq \sum_{j=0}^{k-1} [N(A_0)]^j[Y_1, \cdots, Y_n].$$

Since A_0 is noetherian, there is a k such that $[N(A_0)]^k=0$, and therefore $A[X_1, \cdots, X_n]=A[Y_1, \cdots, Y_n]$. Now we claim that the Y_i are algebraically independent over A. For if $c(Y_1, \cdots, Y_n)=0$ is an equation of algebraic dependence and A_1 is the subring of A generated by A_0 and the coefficients of c, we have

$$A_1[X_1, \cdots, X_n] = A_1[Y_1, \cdots, Y_n].$$

Since the X_i are algebraically independent over A_1 there is an A_1-homomorphism

$$A_1[X_1, \cdots, X_n] \xrightarrow{\tau} A_1[Y_1, \cdots, Y_n]$$

which takes X_i to Y_i. Then $c(X_1, \cdots, X_n)$ must be in the kernel of τ. Since $A_1[Y_1, \cdots, Y_n]=A_1[X_1, \cdots, X_n]$, τ is a homomorphism of $A_1[X_1, \cdots, X_n]$ onto itself. Since any homomorphism of a noetherian ring onto itself is an isomorphism, τ is an isomorphism and hence all of the coefficients of c are zero. Thus the Y's are algebraically independent over A. Now we have immediately that A is isomorphic to B, for if $\mathfrak{A}$ is the ideal of R generated by $\{Y_1, \cdots, Y_n\}$, then $R/\mathfrak{A}$ is isomorphic to both A and B. This concludes the proof of (3).

In [**AEH**] the following is proved:

If A is a domain which is of transcendence degree one over a field, then A is invariant. A is either strongly invariant or there is a field k such that $A=k[t]=k^{(1)}$.

In the sequel any reference to [**AEH**] is an appeal to this result. This shows that if t is transcendental over the field k, then $k[t]$ is an invariant ring which is not strongly invariant, for $k[t][X]=k[X][t]$ but $k[t]\neq k[X]$.

(5) Remark. We do not know of an example of a commutative ring with identity which is not invariant. However there are simple examples

of noninvariant rings without identity. In fact let G be any abelian group such that G is not isomorphic to $G\oplus G$. Make each of these groups into a ring defining the product of any two elements to be zero. Then $G[X]$ and $(G\oplus G)[Y]$ are each isomorphic to a direct sum of countably many copies of G, hence they are isomorphic.

(6) If A is a noetherian ring then A is invariant if and only if whenever

$$A^{(n)} = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n] = B^{(n)},$$

there is a homomorphism from A onto B.

PROOF. Such a homomorphism can be extended to a mapping of $A[X_1, \cdots, X_n]$ onto $B[Y_1, \cdots, Y_n]$ which would be a homomorphism of $A[X_1, \cdots, X_n]$ onto itself. Since any onto endomorphism of a noetherian ring is an isomorphism, the original homomorphism must have been an isomorphism.

(7) Any finite direct sum of invariant rings is invariant; any finite direct sum of strongly invariant rings is strongly invariant.

PROOF. Let $A=A_1\oplus\cdots\oplus A_k$ and

$$A^{(n)} = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n] = B^{(n)} = R,$$

and let e_i be the identity of A_i. Since e_i is idempotent, it is easy to see that $e_i\in B$ for each i. Thus $Be_1+Be_2+\cdots+Be_k$ is a direct sum decomposition of B and $Ae_i[e_iX_1, \cdots, e_iX_n]=Be_i[e_iY_1, \cdots, e_iY_n]=e_iR$ becomes

$$A_i^{(n)} = A_i[\bar{X}_1, \cdots, \bar{X}_n] = B_i[\bar{Y}_1, \cdots, \bar{Y}_n] = B_i^{(n)}.$$

If each A_i is strongly invariant then A_i is isomorphic to B_i for each i and consequently A is isomorphic to B.

We say that the ring R is of *transcendence degree n over a field F* if R contains F, deg $\text{tr}_F\, R/p\leq n$ for each prime ideal $p\subseteq R$, and there is a prime ideal p for which equality is attained.

(8) Let A be a one dimensional noetherian ring which is of transcendence degree one over a field and suppose A has no nontrivial idempotents. Then if A has at least two primes of height zero, A is strongly invariant.

PROOF. Let $A^{(n)}=A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]=B^{(n)}$. We wish to show $A^*=B^*$. Let $q_1, \cdots, q_k$ be the height zero primes of A. Then since $\mathfrak{A}_i=q_i[X_1, \cdots, X_n]$ is also of height zero, there are primes $p_1, \cdots, p_k$ of B such that $p_i[Y_1, \cdots, Y_n]=q_i[X_1, \cdots, X_n]$ and the p_i are precisely the height zero primes of B. Thus we have a diagram:

$$\begin{array}{ccc} A[X_1, \cdots, X_n] = & B[Y_1, \cdots, Y_n] = R, \\ \downarrow & \downarrow \\ A/q_i[\bar{X}_1, \cdots, \bar{X}_n] = & B/p_i[\bar{Y}_1, \cdots, \bar{Y}_n] = R/\mathfrak{A}_i \end{array}$$

and, to see that $A^*=B^*$, it is sufficient to see that $A/q_i=B/p_i$ in $R/\mathfrak{A}_i$ for each i. Since A has no nontrivial idempotents, A cannot be represented as a direct sum and thus q_1 and $b=\prod_{i=2}^{k} q_i$ are contained in a common maximal ideal. For if not we would have $A=A/q_1^t \oplus A/b^t$ whenever $(N(A))^t=0$. Let M be a maximal ideal of A such that $q_1+b \leqq M$. Since M is prime, at least one of the q_i $(i \neq 1)$ is contained in M, say $q_2 \subset M$. Now $q_1+q_2 \leqq M$ implies $q_1[X_1, \cdots, X_n]$, $q_2[X_1, \cdots, X_n] \subseteq M[X_1, \cdots, X_n]$, and therefore $p_1+p_2 \subseteq M[X_1, \cdots, X_n] \cap B=N$. Since p_1 and p_2 are distinct, p_1+p_2 is not of height zero and therefore N is a maximal ideal of B such that $N[Y_1, \cdots, Y_n]=M[X_1, \cdots, X_n]$. Now in

$$R_1 = A/q_1[\bar{X}_1, \cdots, \bar{X}_n] = (A/q_1)^{(n)} = B/p_1[\bar{Y}_1, \cdots, \bar{Y}_n] = (B/p_1)^{(n)}$$

we have that either $A/q_1=B/p_1$ or each is of the form $k[T]=k^{(1)}$ for some field k [**AEH**]. But even in the latter case, they are equal. For each is a P.I.D., so set $M/q_1=(a)$ and $N/p_1=(b)$. Then $aR_1=bR_1$ and so a and b differ only by a unit of R_1. Thus $a \in A/q_1 \cap B/p_1$. Since a generates a nonzero prime of A/q_1, it is transcendental over k. Thus it is a transcendence base for both A/q_1 and B/p_1 over k. But each of A/q_1 and B/p_1 is the algebraic closure of $k[a]$ in R_1, and hence $A/q_1=B/p_1$. Since we could have chosen any prime of height zero to be q_1, it follows that $A^*=B^*$ and we have completed the argument.

(9) REMARK. In case A is a reduced affine ring over a field, the previous argument has a geometric interpretation. One views A as the coordinate algebra of an affine curve Γ_A. Then $A[X]$ is the coordinate algebra of the cylinder C_A over Γ_A (see Figure 1).

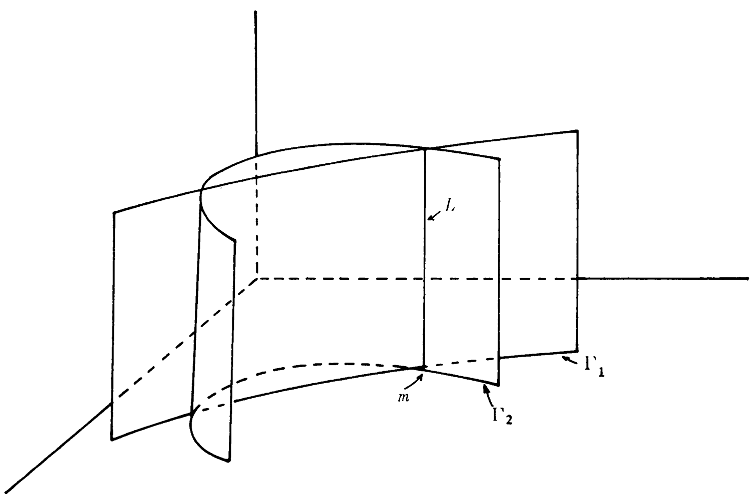

FIGURE 1

If Γ_A is not irreducible, then it is the union of a finite number of irreducible components $\Gamma_1, \cdots, \Gamma_k$ where the Γ_i's correspond to the height zero primes q_i. The assumption that A has no nontrivial idempotents is equivalent to the fact that our curve Γ_A is connected in the Zariski topology. In the proof of (8) the maximal ideal M such $q_1+q_2\subseteq M$ corresponds to a point m on the intersection of the components Γ_1 and Γ_2. The assumption $A[X]\cong B[Y]$ corresponds to the existence of a biregular mapping ϕ from C_A onto a cylinder C_B over some curve Γ_B. With reference to Figure 1 the proof essentially shows that lines such as L on C_A (L corresponds to $M[X]$) must be taken to similar lines on C_B by ϕ (our statement $M[X]=N[Y]$). The proof then can be interpreted as saying that the necessity of lines such as L going onto similar lines imparts so much rigidity that the biregular mapping must essentially take Γ_A onto Γ_B.

(10) Let A be a noetherian ring of transcendence degree one over a field and suppose A has a unique minimal prime p such that $A/p=k[t']=k^{(1)}$. Then A is invariant. Moreover the following conditions are equivalent:

(1) A is strongly invariant.

(2) A is not of the form $A_0[X]=A_0^{(1)}$ where A_0 is a local artinian ring.

(3) A has an embedded prime divisor of (0).

PROOF. First we show the equivalence of conditions (1), (2) and (3) above.

(3)$\Rightarrow$(1). Let p be the unique minimal prime of A. We must show that if $R=A^{(n)}=A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]=B^{(n)}$ then $A^*=B^*$. Let M be an embedded prime divisor of zero in A. Then M is maximal, so there is a maximal ideal $N\subset B$ such that $N[Y_1, \cdots, Y_n]=M[X_1, \cdots, X_n]$. This is because $M[X_1, \cdots, X_n]$ is an associated prime of (0) in R. For if $N_1, \cdots, N_k$ are the associated primes of (0) in B, then $\{N_iR\}_{i=1}^k$ are the primes of (0) in R. By the uniqueness of these primes, $M[X_1, \cdots, X_n]=N_i[Y_1, \cdots, Y_n]$ for some i. Now we go to

$$A^{*(n)} = A^*[\bar{X}_1, \cdots, \bar{X}_n] = B^*[\bar{Y}_1, \cdots, \bar{Y}_n] = B^{*(n)}$$

and use almost exactly the same argument which concludes (8).

(1)$\Rightarrow$(2). This is clear for $A_0[X]=A_0^{(1)}$ is not strongly invariant since $A_0[X][Y]=A_0[Y][X]$.

(2)$\Rightarrow$(3). We must show that if A has no embedded prime divisors of (0), then A is of the form $A_0[X]=A_0^{(1)}$. Let

$$A \xrightarrow{\sigma} A/p=k[t'] = k^{(1)}$$

be the natural map and let $I=\sigma^{-1}(k)$. Since p is the nil radical of I, I is a local ring with maximal ideal p. Thus I is a complete local ring. By Cohen's structure theorem of complete local rings [**ZS**, Theorem 27,

p. 304], I contains a field k_0 which is naturally isomorphic to k. Thus $k_0 \subseteq A$ and $\sigma(k_0)=k$. We now identify k_0 and k.

Choose a preimage t of t' in A. If $\lambda_1, \cdots, \lambda_r$ generate p, we have $k[\lambda_1, \cdots, \lambda_r][t]=A$. The element t is, in fact, transcendental over $k[\lambda_1, \cdots, \lambda_r]=A_0$. For consider the homomorphism

$$k[\lambda_1, \cdots, \lambda_r][X] \xrightarrow{\phi} k[\lambda_1, \cdots, \lambda_r][t] = A.$$

We claim this is an isomorphism. Let $K=$kernel ϕ. Then K must be contained in $N(A_0)[X]=(\lambda_1, \cdots, \lambda_r)[X]$. If not, it would follow that there is a $j(t)\subseteq p$ where j is a nonzero polynomial with coefficients in k. But since $A/p=k[t']$ and t' is transcendental over k, this is impossible. We suppose there is a least integer m such that the kernel of the map σ'' in the following commutative diagram is zero. Here the rows are exact.

$$\begin{array}{ccccccc}
0 \longrightarrow K & \longrightarrow & k[\lambda_1, \cdots, \lambda_r][X] & \xrightarrow{\sigma} & k[\lambda_1, \cdots, \lambda_r][t] = A \\
 & & \downarrow & & \downarrow \\
0 \longrightarrow K' & \longrightarrow & k[\lambda_1', \cdots, \lambda_r'][X] & \xrightarrow{\sigma'} & k[\lambda_1', \cdots, \lambda_r'][t'] = A/(N(A))^{m+1} \\
 & & \downarrow & & \downarrow \\
0 \longrightarrow 0 & \longrightarrow & k[\lambda_1'', \cdots, \lambda_r''][X''] & \xrightarrow{\sigma''} & k[\lambda_1'', \cdots, \lambda_r''][t''] = A/(N(A))^{m}
\end{array}$$

We claim $K'=0$ (for convenience we assume $N(A)^{m+1}=0$). Now $N(A)^m$ is a finite module over $A/N(A)=k[t']$. Moreover this is a torsion free module since if for some $n\in N(A)^m, f\in k[t'], fn=0$ then $f(t)$ would be a zero divisor in A. However A has no embedded primes of zero, and it follows that $f(t)\in p$. This implies $f(t')=0$. Thus there is a free basis for $N(A)^m$ over $A/N(A)$, say $\xi_1, \cdots, \xi_s$. Let $h\in K'$. Then $h=\sum \xi_i h_i(X)$ by the minimality of m. But $\sigma'(h)=\sum \xi_i h_i(t')=0$ implies $h_i=0$ for each i and thus $h=0$. Thus σ' is an isomorphism in contradiction of the minimality of m. It follows then that σ is an isomorphism modulo every power of $N(A)$. But for a suitably high power, this is zero. Thus σ is an isomorphism, and hence (2)$\Rightarrow$(3).

To complete our proof we must show that $A_0[X]=A_0^{(1)}$ is invariant if A_0 is an artinian local ring. Suppose $A_0=k[\lambda_1, \cdots, \lambda_r]$ where the λ_i are nilpotent and let $A=A_0[X]$. If $R=A^{(n)}=A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]=B^{(n)}$, then we may assume $B^*=k[t']=k^{(1)}$ since $k[X]$ is invariant [**AEH**]. Let t be a preimage of t' and

$$(*) \qquad \lambda_i = \eta_i + \sum Y_i g_i(Y_1, \cdots, Y_n) \in B[Y_1, \cdots, Y_n].$$

Then since $N(R)=(\lambda_1, \cdots, \lambda_r)\ R$ it follows that $\{\eta_1, \cdots, \eta_r\}$ generates $N(B)$ and $B=k[\eta_1, \cdots, \eta_r, t]$. If h is a polynomial with coefficients in

k and $h(\lambda_1, \cdots, \lambda_r) = 0$, then substituting from (*) we see $h(\eta_1, \cdots, \eta_r) = 0$. Thus there is a well defined k-homomorphism $k[\lambda_1, \cdots, \lambda_r] \to k[\eta_1, \cdots, \eta_r]$ which takes $\lambda_i \to \eta_i$. This can be extended to a homomorphism of A onto B. Thus A is invariant by (6).

(11) THEOREM. *If A is a one dimensional affine ring over a field k, then A is invariant. Moreover A is strongly invariant unless A can be expressed in the form $A_1 \oplus A_0[X]$ where A_0 is a local artinian ring and X is an indeterminate over A_0.*

PROOF. Let $e_1, \cdots, e_r$ be a maximal set of pairwise orthogonal idempotents of A. Then $A = Ae_1 \oplus \cdots \oplus Ae_r$. Now apply (7), (8) and (10) to complete the argument.

(12) REMARK. Our Definitions (1) and (2) of invariant and strongly invariant could be taken to define n-invariant and n-strongly invariant where n is the number of variables. We do not know if it is possible for a ring to be n-invariant and not m-invariant for different integers m and n. In particular does 1-invariant imply n-invariant?

We close with an example of a strongly invariant, one dimensional affine ring A such that there is an isomorphism of polynomial rings $\sigma: A[X_1, \cdots, X_n] \to B[Y_1, \cdots, Y_n]$, yet $\sigma(A) \neq B$. Let k be a field of characteristic 2 and let $A = k[a^2, a^3, \theta]/(\theta^2)$. If $A^{(n)} = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n] = B^{(n)}$ for some ring B, then $A^* = B^*$ since $A^* = k[a^2, a^3]$ is a strongly invariant ring **[AEH]**. Now let X be an indeterminate over A and set $B = k[a^2, a^3 + \theta X, \theta]$. Then $A[X] = B[X]$ and $A \neq B$. However we must show that X is an indeterminate over B. If not, suppose $b_0 + b_1X + \cdots + b_sX^s = 0$ with $b_i \in B$. Then since $A^* = B^*$, we must have θ divides each b_i, say $b_i = \theta g_i(a^2, a^3 + \theta X, \theta)$. We have then

$$0 = \sum_{i=0}^{s} \theta g_i(a^2, a^3 + \theta X, \theta) X^i = \sum_{i=0}^{s} \theta g_i(a^2, a^3, \theta) X^i$$

since $\theta^2 = 0$. Hence $\theta g_i(a^2, a^3, 0) = 0$ for each i since X is transcendental over A. Therefore $g_i(T, Y) \equiv 0 \bmod (T^3 - Y^2)$ so

$$g_i(T, Y, Z) = (T^3 - Y^2)h(T, Y, Z) + Zq(T, Y, Z)$$

so

$$g_i(a^2, a^3 + \theta X, \theta) = [(a^2)^3 - (a^3 + \theta X)^2]h + \theta q = \theta q.$$

Hence $b_i = \theta g_i(a^2, a^3 + \theta X, \theta) = \theta^2 q = 0$ which shows X is transcendental over B.

ADDED IN PROOF. M. Hochster has recently given an elegant example of an integral domain which is not invariant. Hochster's example is a four dimensional affine ring over the field of real numbers.

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