

CONTINUOUS DEPENDENCE ON A IN THE D_1AD_2 THEOREMS

RICHARD SINKHORN

ABSTRACT. It has been shown by Sinkhorn and Knopp and others that if A is a nonnegative square matrix such that there exists a doubly stochastic matrix B with the same zero pattern as A , then there exists a unique doubly stochastic matrix of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals. Sinkhorn and Knopp have also shown that if A has at least one positive diagonal, then the sequence of matrices obtained by alternately normalizing the row and column sums of A will converge to a doubly stochastic limit. It is the intent of this paper to show that D_1AD_2 and/or the limit of this iteration, when either exists, is continuously dependent upon the matrix A .

Introduction. An $N \times N$ matrix $A = (a_{ij})$ is said to be nonnegative if every $a_{ij} \geq 0$. For such a matrix A we write $A \geq 0$.

An $N \times N$ matrix $A = (a_{ij})$ is said to be doubly stochastic if $A \geq 0$ and if $\sum_{k=1}^N a_{ik} = \sum_{k=1}^N a_{kj} = 1$ for all i and j . The set of $N \times N$ doubly stochastic matrices is denoted by Ω_N .

We say that the $N \times N$ nonnegative matrices A and B have the same pattern if $a_{ij} = 0 \Leftrightarrow b_{ij} = 0$. We say that the $N \times N$ nonnegative matrix A has a subpattern of an $N \times N$ nonnegative matrix B if $a_{ij} = 0 \Rightarrow b_{ij} = 0$. If A is an $N \times N$ nonnegative matrix such that there exists a $B \in \Omega_N$ with the same pattern as A , we say that A has doubly stochastic pattern. The set of all $N \times N$ nonnegative matrices with doubly stochastic pattern is denoted by $\mathcal{P}(\Omega_N)$. If A is an $N \times N$ nonnegative matrix and if there exists a $B \in \Omega_N$ such that A has a subpattern of B , we say that A has doubly stochastic subpattern. The set of all $N \times N$ nonnegative matrices with doubly stochastic subpattern is denoted by $\mathcal{S}(\Omega_N)$. Observe that $\mathcal{P}(\Omega_N) \subseteq \mathcal{S}(\Omega_N)$.

We denote by S_N the set of all permutations of $1, \dots, N$. If A is an $N \times N$ matrix and $\sigma \in S_N$, the set of elements $a_{1\sigma(1)}, \dots, a_{N\sigma(N)}$ is called a diagonal of A . If every $a_{i\sigma(i)} > 0$, we say that the diagonal is positive. In case σ is the identity permutation, we call the diagonal the main diagonal of A .

Received by the editors November 6, 1970.

AMS 1970 subject classifications. Primary 15A51, 15A15; Secondary 15A21.

Key words and phrases. Doubly stochastic matrix, permanent, doubly stochastic pattern, doubly stochastic subpattern.

© American Mathematical Society 1972

If A is an $N \times N$ matrix, we define the permanent of A , $\text{per } A$, by

$$\text{per } A = \sum_{\sigma \in S_N} \prod_{i=1}^N a_{i\sigma(i)}.$$

In [2], Sinkhorn and Knopp show that if $A \in \mathcal{P}(\Omega_N)$, there exists a unique matrix of the form $D_1 A D_2 \in \Omega_N$ where D_1 and D_2 are diagonal matrices with positive main diagonals. They show that $D_1 A D_2$ is the limit of a sequence of matrices obtained by alternately scaling the rows and columns of A . They show in fact that this matrix sequence will converge to a limit in Ω_N if and only if $A \in \mathcal{S}(\Omega_N)$. The limit has the form $D_1 A D_2$, however, only if $A \in \mathcal{P}(\Omega_N)$.

It is the intent of this paper to show that $D_1 A D_2$ is a continuous function of A on $\mathcal{P}(\Omega_N)$ and that the limit of the iteration is a continuous function of A on $\mathcal{S}(\Omega_N)$. The following result of Sinkhorn and Knopp [3] is the main tool in the development.

THEOREM 1. *Distinct $N \times N$ doubly stochastic matrices A and B do not have proportional corresponding diagonal products, i.e. there is no $k > 0$ such that for each $\sigma \in S_N$, $\prod_{i=1}^N a_{i\sigma(i)} = k \prod_{i=1}^N b_{i\sigma(i)}$.*

The following celebrated theorem of G. Birkhoff may be employed to show that $\text{per } A > 0$ for any $A \in \Omega_N$. For a proof see [1, p. 98].

THEOREM 2. *The set of all $N \times N$ doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices.*

Results and consequences.

THEOREM 3. *Suppose that $A \geq 0$ is an $N \times N$ matrix such that $\text{per } A > 0$. Suppose that for each positive integer n and $k=1, 2$, $A_k(n) \geq 0$ is an $N \times N$ matrix such that for every permutation $\sigma \in S_N$, $\lim_{n \rightarrow \infty} \prod_{i=1}^N a_{k i \sigma(i)}(n) = \prod_{i=1}^N a_{i \sigma(i)}$. If for each positive integer n and $k=1, 2$, $E_k(n)$ and $F_k(n)$ are $N \times N$ diagonal matrices with positive main diagonals such that*

$$\lim_{n \rightarrow \infty} E_k(n) A_k(n) F_k(n) = P_k \in \Omega_N,$$

then necessarily $P_1 = P_2$.

PROOF. For $k=1, 2$, put $K_k = \text{per } P_k / \text{per } A$. Since each $P_k \in \Omega_N$, $\text{per } P_k > 0$ and so each $K_k > 0$. It is seen that for each such value of k , $\text{per } A_k(n) \rightarrow \text{per } A$, and thus

$$\lim_{n \rightarrow \infty} \text{per } E_k(n) F_k(n) = \lim_{n \rightarrow \infty} \frac{\text{per } E_k(n) A_k(n) F_k(n)}{\text{per } A_k(n)} = \frac{\text{per } P_k}{\text{per } A} = K_k.$$

Let $\Delta_A \subseteq S_N$ denote those permutations σ such that $a_{1\sigma(1)}, \dots, a_{N\sigma(N)}$ is a positive diagonal in A . Since $\text{per } A > 0$, $\Delta_A \neq \emptyset$.

Put $P_k=(p_{kij})$ for $k=1, 2$. Then for either value of k and any $\sigma \in \Delta_A$,

$$\prod_{i=1}^N \frac{p_{ki\sigma(i)}}{a_{i\sigma(i)}} = \lim_{n \rightarrow \infty} \frac{[\text{per } E_k(n)F_k(n)] \prod_{i=1}^N a_{ki\sigma(i)}(n)}{\prod_{i=1}^N a_{i\sigma(i)}} = K_k,$$

where $A_k(n)=(a_{kij}(n))$.

From

$$\begin{aligned} \text{per } P_k &= \sum_{\sigma \in S_N} \prod_{i=1}^N p_{ki\sigma(i)} = \sum_{\sigma \in \Delta_A} \prod_{i=1}^N p_{ki\sigma(i)} + \sum_{\sigma \in S_N - \Delta_A} \prod_{i=1}^N p_{ki\sigma(i)} \\ &= K_k \sum_{\sigma \in \Delta_A} \prod_{i=1}^N a_{i\sigma(i)} + \sum_{\sigma \in S_N - \Delta_A} \prod_{i=1}^N p_{ki\sigma(i)} \\ &= K_k \text{per } A + \sum_{\sigma \in S_N - \Delta_A} \prod_{i=1}^N p_{ki\sigma(i)} = \text{per } P_k + \sum_{\sigma \in S_N - \Delta_A} \prod_{i=1}^N p_{ki\sigma(i)}, \end{aligned}$$

we see that $\prod_{i=1}^N p_{ki\sigma(i)}=0$ for each $\sigma \in S_N - \Delta_A$, $k=1, 2$. Since $\prod_{i=1}^N a_{i\sigma(i)}=0$ for each $\sigma \in S_N - \Delta_A$, it follows that $\prod_{i=1}^N p_{1i\sigma(i)}=K_1 \prod_{i=1}^N a_{i\sigma(i)}=(K_1/K_2) \prod_{i=1}^N p_{2i\sigma(i)}$ for all $\sigma \in S_N$. Thus, by Theorem 1, $P_1=P_2$.

COROLLARY 1. *Let $A \geq 0$ be an $N \times N$ matrix such that $\text{per } A > 0$. Put $\bar{A}=(\bar{a}_{ij})$ where $\bar{a}_{ij}=a_{ij}$ if a_{ij} lies on at least one positive diagonal in A , and $\bar{a}_{ij}=0$ otherwise. Let D_1 and D_2 be diagonal matrices with positive main diagonals such that $D_1\bar{A}D_2 \in \Omega_N$. Then the limit of the iteration of alternately normalizing the row and column sums of A is equal to $D_1\bar{A}D_2$.*

PROOF. Let the n th term of the iteration be denoted by $E_1(n)A_1(n)F_1(n)$ where $A_1(n) \equiv A$. Also put $E_2(n)=D_1$, $A_2(n)=\bar{A}$, and $F_2(n)=D_2$ for all n . It follows from the Sinkhorn-Knopp result [2] that $\lim_{n \rightarrow \infty} E_1(n)A_1(n)F_1(n)$ exists. Since $\prod_{i=1}^N a_{ki\sigma(i)}(n)=\prod_{i=1}^N a_{i\sigma(i)}=\prod_{i=1}^N \bar{a}_{i\sigma(i)}$ for any $\sigma \in S_N$, Theorem 3 shows that this limit is in fact $D_1\bar{A}D_2$.

COROLLARY 2. *Let $A \geq 0$ be an $N \times N$ matrix such that $\text{per } A > 0$. For any $\epsilon > 0$ suppose that $A(\epsilon) \geq 0$ is an $N \times N$ matrix such that*

$$\lim_{\epsilon \downarrow 0} \prod_{i=1}^N a_{i\sigma(i)}(\epsilon) = \prod_{i=1}^N a_{i\sigma(i)}$$

for every $\sigma \in S_N$. Let \bar{A} , D_1 , and D_2 be as in Corollary 1, and let $\bar{A}(\epsilon)$, $D_1(\epsilon)$, and $D_2(\epsilon)$ be the corresponding matrices for $A(\epsilon)$ whenever $\text{per } A(\epsilon) > 0$. Then

$$\lim_{\epsilon \downarrow 0} D_1(\epsilon)\bar{A}(\epsilon)D_2(\epsilon) = D_1\bar{A}D_2.$$

PROOF. Observe that $D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)$ exists for ε sufficiently small since for such values of ε , per $A(\varepsilon) > 0$.

Since Ω_N is compact, the set $\{D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)\}$ is bounded and therefore has at least one limit point as $\varepsilon \downarrow 0$. Of course all the limit points of this set are doubly stochastic. Moreover, since

$$\lim_{\varepsilon \downarrow 0} \prod_{i=1}^N \bar{a}_{i\sigma(i)}(\varepsilon) = \lim_{\varepsilon \downarrow 0} \prod_{i=1}^N a_{i\sigma(i)}(\varepsilon) = \prod_{i=1}^N a_{i\sigma(i)} = \prod_{i=1}^N \bar{a}_{i\sigma(i)}$$

for all $\sigma \in S_N$, any two convergent subsequences of $\{D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)\}$ must have the same limit by Theorem 3. Thus the set $\{D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon)\}$ has exactly one limit point as $\varepsilon \downarrow 0$ and so $\lim_{\varepsilon \downarrow 0} D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon) = P$ exists. Clearly $P \in \Omega_N$.

Put $E_1(n) = D_1(1/n)$, $A_1(n) = \bar{A}(1/n)$, and $F_1(n) = D_2(1/n)$ and put $E_2(n) = D_1$, $A_2(n) = \bar{A}$, and $F_2(n) = D_2$ for $n = 1, 2, \dots$. By Theorem 3,

$$\lim_{n \rightarrow \infty} E_1(n)A_1(n)F_1(n) = P = D_1\bar{A}D_2 = \lim_{n \rightarrow \infty} E_2(n)A_2(n)F_2(n).$$

Whence

$$\lim_{\varepsilon \downarrow 0} D_1(\varepsilon)\bar{A}(\varepsilon)D_2(\varepsilon) = D_1\bar{A}D_2.$$

Corollaries 3 and 4 which follow are immediate consequences of Corollaries 1 and 2.

COROLLARY 3. *The limit of the iteration of alternately normalizing the row and column sums of an $N \times N$ matrix A is a continuous function of A on $\mathcal{S}(\Omega_N)$.*

COROLLARY 4. *For each $A \in \mathcal{P}(\Omega_N)$ there is a unique matrix $D_1AD_2 \in \Omega_N$ where D_1 and D_2 are diagonal matrices with positive main diagonals. The map $A \rightarrow D_1AD_2$ is continuous on $\mathcal{P}(\Omega_N)$.*

The author is most grateful for many enlightening discussions with Paul Knopp and Mark Hedrick concerning this presentation.

REFERENCES

1. Marvin Marcus and Henryk Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, Mass., 1964. MR 29 #112.
2. Richard Sinkhorn and Paul Knopp, *Concerning nonnegative matrices and doubly stochastic matrices*, Pacific J. Math. 21 (1967), 343-348. MR 35 #1617.
3. ———, *Problems involving diagonal products in nonnegative matrices*, Trans. Amer. Math. Soc. 136 (1969), 67-75. MR 38 #2151.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004