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AUTOMORPHISM OF A FINITE GROUP SCALAR ON THE COSETS OF A SUBGROUP

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ABSTRACT Let G be a finite group, σ an automorphism of G, M a σ-invariant subgroup of G, and n a fixed integer. If $\sigma(g)\in g^nM$ for all $g\in G$ then there exists a σ-invariant normal subgroup K of G, contained in M, with $\sigma(g)\in g^nK$ for all $g\in G$.

1. **Introduction.** The purpose of this paper is to prove the following result.

THEOREM. *Let G be a finite group, σ an automorphism of G, M a σ-invariant subgroup of G, and n a fixed integer. If $\sigma(g)\in g^nM$ for all $g\in G$ then there exists K a σ-invariant subgroup of M, $K\lhd G$, and $\sigma(g)\in g^nK$ for all $g\in G$.*

It follows from the theorem that $\bar{G}=G/K$ is n-abelian; i.e., $(ab)^n=a^nb^n$ for all $a, b\in G$. Such groups have been classified by Alperin in [**1**].

It is well known that for $n=1$, K can be taken to be $\langle[G, \sigma]\rangle$.

When M is the trivial subgroup, we refer to σ as a scalar automorphism. In this case $[G, \sigma]\subseteq Z(G)$, and σ is a fixed point free automorphism iff $(n-1, o(G))=1$ (see [**2**]).

Throughout this paper G will stand for a finite group and σ for an automorphism of G. We will find it convenient to regard G and $\langle\sigma\rangle$ as embedded in the semidirect product of G by $\langle\sigma\rangle$; e.g., $[g, \sigma]=g^{-1}\sigma^{-1}g\sigma$. The rest of our notation is standard (see [**4**] or [**5**]).

2. **Preliminary results.**

LEMMA 1. *Let σ be an automorphism of G of prime order p. Suppose $[x, \sigma, \sigma]=e$ for all $x\in G$ such that $x^p=e$. Suppose furthermore that for some $y\in G$, $[y, \sigma, \sigma]=e$, yet $[y, \sigma]\neq e$. Then $O_p(G)\neq E$.*

PROOF. Denote $C_G(\sigma)$ by H. Let $y\in G$ such that $e\neq z=[y, \sigma]\in H$. Then $z^p=e$, $z_1=[y^{-1}, \sigma]^{-1}=[y, \sigma]^{y^{-1}}$, and $z_1\in H$.

Let $t\in H$, $t^p=e$. Then, as $[t^y, \sigma, \sigma]=e$, we have, from $e=[t^y, \sigma\sigma^{-1}]=[t^y, \sigma^{-1}][t^y, \sigma]$,

$$[t^y, \sigma]^{-1} = [t^y, \sigma^{-1}].$$

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Since $t^{yz}=t^y[t^y, \sigma]$,

$$[t^{yz}, \sigma^{-1}] = [t^y, \sigma]^{-1}$$

and so $[t^y, \sigma]$ commutes with z. Thus, z_1 commutes with $[t, z_1]$. Let $t=z_1^{-h}$, $h\in H$. Then the iterated commutator $[h, 3z_1]=e$. In other words, z_1 is a left Engel element of H and so by a theorem of Baer [**5**, p. 212], $z_1\in O_p(H)$. If y^{-1} is substituted for y in the preceding argument we get $z\in O_p(H)$.

Let $v\in Z(O_p(H))$, $o(v)=p$. If $x\in G$ and $o(x)=p$, then $v^x\in H$. Let $y=v^{-g}$, $g\in G$. Then, $[y, v]=[v^{-g}, v]=[g, 2v]^{v^{-1}}$ is an element of H. Since $v\in O_p(H)$, $[g, kv]=e$ for some positive integer k. Thus v is a left Engel element of G; hence, $v\in O_p(G)\neq E$.

Definition and Remark. Let n be a fixed integer and G a finite group. Define $\mathcal{N}(G)=\langle g | g\in G \text{ and } g^n=e\rangle$. $\mathcal{N}(G)$ is a characteristic subgroup of G. Let $\mathcal{N}_0=\mathcal{N}(G)$ and define $\mathcal{N}_i$ inductively as the preimage of $\mathcal{N}(G/\mathcal{N}_{i-1})$ in G. Let $\mathcal{N}_\infty=\bigcup_{i=0}^{\infty}\mathcal{N}_i$. Then $o(G/\mathcal{N}_\infty)$ and n are coprime.

In the sequel, we will let n be a fixed integer, and M a σ-invariant subgroup of G such that $\sigma(g)=g^n m(g)$, $m(g)\in M$, for all $g\in G$.

We begin with an inheritance-type lemma the proof of which is direct.

Lemma 2. *Let F and H be σ-invariant subgroups of G, $F\lhd H$, and $\bar\sigma$ the automorphism induced by σ on $\bar H=H/F$. Then*

$$\overline{M\cap H} = (M\cap H)F/F$$

is $\bar\sigma$-invariant and

$$\bar\sigma(\bar h) = \bar h^n\overline{m(h)}, \qquad \overline{m(h)}\in\overline{M\cap H},$$

for all $\bar h\in\bar H$.

Lemma 3. $[M, \sigma]^G\mathcal{N}_\infty\subseteq M$.

Proof. Let $g\in G$, $t\in M$.

$$\sigma(g) = g^n m(g),$$
$$\sigma(t^{-1}gt) = t^{-1}g^n t m(g^t) = \sigma(t)^{-1}\sigma(g)\sigma(t) = \sigma(t)^{-1}g^n m(g)\sigma(t).$$

Thus, $[g^n, \sigma(t)t^{-1}]\in M$. If n and $o(G)$ are coprime then $[G, [\sigma, t]]\subseteq M$ for all $t\in M$; thence, $[\sigma, M]^G\subseteq M$. Now if $\mathcal{N}_\infty\subseteq M$, then by Lemma 2 we may translate the situation to $G/\mathcal{N}_\infty$; in which case we are finished. We may assume that there exists $g\in G$, $g\neq e$ and $g^n=e$. Then $\sigma(g)\in g^nM=M$ and so $g\in M$; i.e., $\mathcal{N}_0\subseteq M$. On applying induction to $o(G)$, and since $o(G/\mathcal{N}_0)<o(G)$, $\mathcal{N}_\infty\subseteq M$; the proof is finished.

Lemma 4. *Suppose $\sigma|_M=1$. Then*

(i) $G^{n^{o(\sigma)}-1}\subseteq M$,

(ii) g^{n-1} *commutes with* $m(g)$ *for all* $g\in G$,

(iii) *for any subgroup H of G, $\langle[H, \sigma]\rangle$ is σ-invariant.*

PROOF. Part (i) follows from $\sigma^i(g)\in g^{n^i}M$, i integer ≥ 0. As for part (ii), $\sigma(g^{-1})=g^{-n}m(g^{-1})=m(g)^{-1}g^{-n}$ implies $g^n m(g) g^{-n}=m(g^{-1})^{-1}$. On expanding each term of $\sigma(g)=\sigma(g^{1-1/n})\sigma(g^{1/n})$ and employing the previous equation we get $a=m(g)^{g^{-1}}\in M$; now (ii) follows from $\sigma(a)=a$. To prove part (iii) it suffices to observe, $[g, 2\sigma]=[g^{n-1}, \sigma]$.

LEMMA 5. *Suppose* $o(\sigma)=p$, $p\in\pi(G)$. *Let* $n^p\equiv 1$ *modulo* $\exp(G)$, *and* $[M, \sigma]=E$. *Then* $[G, \sigma]\subseteq N_G(P)$, *where* $P\in\mathrm{Syl}_p(G)$.

PROOF. We proceed by induction on $o(G)$. We may assume that for some $x\in G$ such that $o(x)=p$, $x\notin M$. Then, as $n\equiv 1 \bmod p$, we get $[x, 2\sigma]=e$ and $[x, \sigma]\neq e$. Thus by Lemma 1, $O_p(G)\neq E$. The conclusion follows by applying the inductive hypothesis to $G/O_p(G)$.

3. **Proof of the theorem.** Let G be a counterexample of minimal order. Clearly, the only subgroup of M normal in G is E. From Lemmas 3 and 4 we conclude that $[M, \sigma]=E$, $(n, o(G))=1$, $n^{o(\sigma)}\equiv 1$ modulo $\exp(G)$. In addition, we may regard M to be a maximal subgroup of G. There are proper nontrivial normal subgroups in G. This is certainly true if $(o(\sigma), o(G))\neq 1$, as is implied by Lemma 5. On the other hand if $o(\sigma)$ and $o(G)$ are coprime then $o(G)$ is odd; otherwise, if $x\in G$ and $o(x)=2$, then $x^n=x$, $[x, \sigma]^{o(\sigma)}=e$, and $x\in M$. In this case, by the Odd Order paper [3], G is solvable.

Let H be a minimal nontrivial normal subgroup in G; therefore H is semisimple and $H\neq G$. Then $G=HM$ and $R=\langle[H, \sigma]\rangle=\langle[G, \sigma]\rangle$ is σ-invariant nontrivial normal in G. So, $G=RM$.

Suppose $G\neq R$ and let $M_0=R\cap M$. Furthermore, suppose M_0 is trivial. Then σ is fixed point free on R and is scalar n on R. Hence R is an elementary abelian q-group for some prime q. Let $x\in R$, $t\in M$. Then $[xt, \sigma]=(xt)^{n-1}m(xt)$, and $[xt, \sigma]=[x, \sigma]^t=(x^{n-1})^t$. Since $(xt)^{n-1}=f(x)t^{n-1}$ where $f(x)\in R$, we conclude $t^{n-1}m(xt)=e$ and $f(x)=(x^{n-1})^t$; so by Lemma 4, t^{n-1} commutes with $(x^{n-1})^t$. Moreover, since $n-1$ and $o(x)$ are coprime, x commutes with t^{n-1}. Hence, R commutes with M^{n-1}; so, M^{n-1} is trivial. Furthermore, from $t^{n-1}=m(xt)^{-1}=e$, we get $\sigma(xt)=(xt)^n$; therefore, $R=[G, \sigma]\subseteq Z(G)$, and so $M\lhd G$ which is impossible.

Hence, $E\neq M_0\lhd M$ and $\sigma(k)\in k^n M_0$ for all $k\in R$. Since $R\neq G$, there exists K_0 a subgroup of M_0, $K_0\neq E$, $K_0\lhd R$, such that σ induces a scalar n on R/K_0. $K_1=K_0^M\subseteq M_0$, $K_1\lhd R$ and so $K_1\lhd G$ which is impossible.

So far then, $G=R=H\sigma(H)$, $H\cap\sigma(H)=E$, elements of H commute with those of $\sigma(H)$, and H is simple nonabelian. Again we make use of the Odd Order paper and let x be an involution in H. Since x commutes with $[x, \sigma]$, $[x, \sigma]^2=e$. Also, since $[x, \sigma]\in M$, H is σ^2-invariant and σ^2 is trivial on H. Consequently, $o(\sigma)=2$; however, using Lemma 5, $R\neq G$. A contradiction is reached and the theorem is established.

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