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A REMARK ON GROUPS WITH THE FIXED POINT PROPERTY

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ABSTRACT. We prove that any group with the fixed point property actually leaves fixed points for measurable actions rather than only jointly continuous actions.

One says a locally compact group, G, has the fixed point property **[2]**–**[4]** if and only if every jointly continuous affine action of G on a compact convex subset, K, of a locally convex topological vector space, E, has a fixed point. A jointly continuous affine action of G on K is a map $(g, x)\to\alpha_g(x)$ of $G\times K\to K$ which is jointly continuous with each α_g affine. There are obviously other fixed point properties one might define by weakening the continuity properties required of the action. Specifically:

DEFINITION. A *weakly measurable* affine action of G on a compact convex subset, K, of a locally convex topological vector space is a representation of G by continuous affine maps of $K\to K$ so that for each $l\in E^*$ and $x\in K$, $g\to l(\alpha_g(x))$ is measurable. We say G has the *strong fixed point property* if every weakly measurable affine action of G on a compact convex subset, K, has a fixed point in K.

We remark, when K is not separable, weak measurability may hold for discontinuous actions as is shown by:

EXAMPLE. Let E be the Hilbert space of all functions on $\boldsymbol{R}$ with $\sum_{x\in\boldsymbol{R}} |f(x)|^2<\infty$, i.e. $f\in E$ is 0 except for a countable set. Topologize E with the weak topology and let K be the unit ball. For $t\in\boldsymbol{R}$ let $(\alpha_t f)(x)=f(x+t)$. It is easy to see α_t is weakly measurable but not continuous.

Our goal here is to note that G has the strong fixed point property if and only if it has the fixed point property. This is actually a very simple consequence of the Greenleaf-Namioka theorem **[3]** on the equivalance of the various notions of amenability.

THEOREM. *The following are equivalent for a locally compact group*, G:

(a) *There is a left invariant mean on* $L^\infty(G)$.

(b) *G has the strong fixed point property.*

(c) *G has the fixed point property.*

(d) *There is a left invariant mean on the functions on G which are bounded and uniformly continuous on the right.*

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PROOF. (b)$\Rightarrow$(c) is trivial. (c)$\Rightarrow$(d), in fact (c)$\Leftrightarrow$(d) is a result of Rickert [**4**]. (d)$\Rightarrow$(a) is the Greenleaf-Namioka theorem. Thus, we need only prove (a)$\Rightarrow$(b). Suppose (a) holds and let $g\to\alpha_g$ be a weakly measurable action of G on K, a compact convex subset of a locally convex space, E. Pick $x\in K$. For each $l\in E^*$, $g\to l(\alpha_g(x))$ is a function in L^∞ (since l is bounded on K). Let m be the left invariant mean on L^∞.

Define $F(l)=m(l(\alpha_g(x)))$. $F(l)$ is linear in l and $\sup_{x\in K} l(x)\geqq F(l)\geqq \inf_{x\in K} l(x)$ for any real linear functional. If we can show $F(l)=l(y)$ for some $y\in E$, it follows from the Hahn-Banach separation theorem that $y\in K$. If we know $y\in K$, then, for any $l\in E^*$, $h\in G$,

$$l(\alpha_h(y)) = (l\circ\alpha_h)(y) = m_g(l(\alpha_h\alpha_g(x))) = m_g(l(\alpha_g(x))) = l(y).$$

Again using the Hahn-Banach theorem, $\alpha_h(y)=y$.

It only remains to prove $F(l)=l(y)$ for some y. By the Mackey-Arens theorem [**1**], we need only show $F(l)$ is continuous when the Mackey topology, $\tau(E^*, E)$, is put on E^*. If $l_\alpha\to l$ in the Mackey topology, $l_\alpha(z)\to l(z)$ uniformly for z in a compact subset of E; in particular uniformly for $z\in K$. Thus $l_\alpha(\alpha_g(x))$ converges to $l(\alpha_g(x))$ in $L^\infty(M)$. Since m is an L^∞ continuous functional, $F(l_\alpha)\to F(l)$. Q.E.D.

We remark that the proof in [**3**] and [**4**] of (d)$\Rightarrow$(c) does not extend to (a)$\Rightarrow$(b) so that the trick of using the Mackey topology is essential.

It is a pleasure to thank Mike Reed for suggesting the example above.

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