

A REMARK ON GROUPS WITH THE FIXED POINT PROPERTY

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ABSTRACT. We prove that any group with the fixed point property actually leaves fixed points for measurable actions rather than only jointly continuous actions.

One says a locally compact group, G , has the fixed point property [2]–[4] if and only if every jointly continuous affine action of G on a compact convex subset, K , of a locally convex topological vector space, E , has a fixed point. A jointly continuous affine action of G on K is a map $(g, x) \rightarrow \alpha_g(x)$ of $G \times K \rightarrow K$ which is jointly continuous with each α_g affine. There are obviously other fixed point properties one might define by weakening the continuity properties required of the action. Specifically:

DEFINITION. A *weakly measurable* affine action of G on a compact convex subset, K , of a locally convex topological vector space is a representation of G by continuous affine maps of $K \rightarrow K$ so that for each $l \in E^*$ and $x \in K$, $g \rightarrow l(\alpha_g(x))$ is measurable. We say G has the *strong fixed point property* if every weakly measurable affine action of G on a compact convex subset, K , has a fixed point in K .

We remark, when K is not separable, weak measurability may hold for discontinuous actions as is shown by:

EXAMPLE. Let E be the Hilbert space of all functions on \mathbf{R} with $\sum_{x \in \mathbf{R}} |f(x)|^2 < \infty$, i.e. $f \in E$ is 0 except for a countable set. Topologize E with the weak topology and let K be the unit ball. For $t \in \mathbf{R}$ let $(\alpha_t f)(x) = f(x+t)$. It is easy to see α_t is weakly measurable but not continuous.

Our goal here is to note that G has the strong fixed point property if and only if it has the fixed point property. This is actually a very simple consequence of the Greenleaf-Namioka theorem [3] on the equivalence of the various notions of amenability.

THEOREM. *The following are equivalent for a locally compact group, G :*

- (a) *There is a left invariant mean on $L^\infty(G)$.*
- (b) *G has the strong fixed point property.*
- (c) *G has the fixed point property.*
- (d) *There is a left invariant mean on the functions on G which are bounded and uniformly continuous on the right.*

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PROOF. (b) \Rightarrow (c) is trivial. (c) \Rightarrow (d), in fact (c) \Leftrightarrow (d) is a result of Rickert [4]. (d) \Rightarrow (a) is the Greenleaf-Namioka theorem. Thus, we need only prove (a) \Rightarrow (b). Suppose (a) holds and let $g \rightarrow \alpha_g$ be a weakly measurable action of G on K , a compact convex subset of a locally convex space, E . Pick $x \in K$. For each $l \in E^*$, $g \rightarrow l(\alpha_g(x))$ is a function in L^∞ (since l is bounded on K). Let m be the left invariant mean on L^∞ .

Define $F(l) = m(l(\alpha_g(x)))$. $F(l)$ is linear in l and $\sup_{x \in K} l(x) \geq F(l) \geq \inf_{x \in K} l(x)$ for any real linear functional. If we can show $F(l) = l(y)$ for some $y \in E$, it follows from the Hahn-Banach separation theorem that $y \in K$. If we know $y \in K$, then, for any $l \in E^*$, $h \in G$,

$$l(\alpha_h(y)) = (l \circ \alpha_h)(y) = m_g(l(\alpha_h \alpha_g(x))) = m_g(l(\alpha_g(x))) = l(y).$$

Again using the Hahn-Banach theorem, $\alpha_h(y) = y$.

It only remains to prove $F(l) = l(y)$ for some y . By the Mackey-Arens theorem [1], we need only show $F(l)$ is continuous when the Mackey topology, $\tau(E^*, E)$, is put on E^* . If $l_\alpha \rightarrow l$ in the Mackey topology, $l_\alpha(z) \rightarrow l(z)$ uniformly for z in a compact subset of E ; in particular uniformly for $z \in K$. Thus $l_\alpha(\alpha_g(x))$ converges to $l(\alpha_g(x))$ in $L^\infty(M)$. Since m is an L^∞ continuous functional, $F(l_\alpha) \rightarrow F(l)$. Q.E.D.

We remark that the proof in [3] and [4] of (d) \Rightarrow (c) does not extend to (a) \Rightarrow (b) so that the trick of using the Mackey topology is essential.

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