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AN ALGEBRAIC CHARACTERIZATION OF DIMENSION

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ABSTRACT. The purpose of this paper is to translate the condition defining Lebesgue covering dimension of a topological space X into a condition on $C(X)$, the ring of continuous real-valued functions on X.

We take the definition of topological dimension given in [2, p. 243]. The definition in [2] is given for completely regular Hausdorff spaces, but applies equally well to arbitrary spaces. Characterizations of dim X in terms of $C(X)$ have been given by Katětov and by the author [1]. For an exposition of Katětov's work the reader is referred to [2, Chapter 16].

Let R be a commutative ring with identity. By a *basis* in R we mean a finite set of elements which generate R. The *order* of a basis is the largest integer n for which there exist $n+1$ members of the basis with nonzero product.

There is a close relation between bases in $C(X)$ and basic covers of X [2, p. 243]. For each basis $\{f_i\}_{i\in I}$ of $C(X)$ we associate the basic cover $\{U_i\}_{i\in I}$ of X, where U_i is defined by $U_i=\{x:f_i(x)\neq 0\}$. Conversely, for each basic cover $\{U_i\}_{i\in I}$ of X, we may associate a basis $\{f_i\}_{i\in I}$ of $C(X)$ where f_i is chosen to satisfy $U_i=\{x:f_i(x)\neq 0\}$. Since $U_{i_1}\cap\cdots\cap U_{i_n}=\varnothing$ if and only if $f_{i_1}\cdots f_{i_n}=0$, it follows that the order of the basic cover $\{U_i\}_{i\in I}$ is the same as the order of the basis $\{f_i\}_{i\in I}$.

If now $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are bases in the ring R, we say that $\{b_j\}_{j\in J}$ is a *refinement* of $\{a_i\}_{i\in I}$ if for each $j\in J$ there is an $i\in I$ such that b_j is a multiple of a_i. The *dimension* of R, denoted by $d(R)$, is here defined to be the least cardinal $\mathfrak{m}$ such that every basis of R has a refinement of order at most $\mathfrak{m}$.

THEOREM. *If X is an arbitrary topological space, then* dim $X=d(C(X))$.

PROOF. Suppose $d(C(X))\leqq n$. Let $\{U_i\}_{i\in I}$ be a basic cover of X and let $\{f_i\}_{i\in I}$ be an associated basis in $C(X)$. By hypothesis, this basis has a refinement $\{g_j\}_{j\in J}$ of order at most n. The basic cover associated with $\{g_j\}_{j\in J}$ is then a refinement of $\{U_i\}_{i\in I}$ of order at most n. Thus dim $X\leqq n$.

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Suppose now that dim $X \leqq n$. Let $\{f_i\}_{i \in I}$ be a basis in $C(X)$ and let $\{U_i\}_{i \in I}$ be the associated basic cover of X. By hypothesis, this cover has a basic refinement $\{V_j\}_{j \in J}$ of order at most n. By Theorem 16.6 of [2], there are zero-sets Z_j which cover X and for which $Z_j \subset V_j$ for $j \in J$. Let $k_j \in C(X)$ satisfy the following: $k_j(x)=0$ if $x \notin V_j$ and $k_j(x)=1$ if $x \in Z_j$. For each $j \in J$, choose $i(j) \in I$ such that $V_j \subset U_{i(j)}$, and let $g_j = k_j f_{i(j)}$. From the construction of g_j it follows that

$$Z_j \subset \{x : g_j(x) \neq 0\} \subset V_j.$$

Since the Z_j's cover X, it follows that $\{g_j\}_{j \in J}$ is a basis of $C(X)$, which clearly refines $\{f_i\}_{i \in I}$. Since $\{V_j\}_{j \in J}$ has order at most n, the family of sets $\{x : g_j(x) \neq 0\}$ also has order at most n, whence the basis $\{g_j\}_{j \in J}$ has order at most n. Thus $d(C(X)) \leqq n$ and the proof is complete.

References

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