AN ALGEBRAIC CHARACTERIZATION OF DIMENSION

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Abstract. The purpose of this paper is to translate the condition defining Lebesgue covering dimension of a topological space $X$ into a condition on $C(X)$, the ring of continuous real-valued functions on $X$.

We take the definition of topological dimension given in [2, p. 243]. The definition in [2] is given for completely regular Hausdorff spaces, but applies equally well to arbitrary spaces. Characterizations of $\dim X$ in terms of $C(X)$ have been given by Katětov and by the author [1]. For an exposition of Katětov's work the reader is referred to [2, Chapter 16].

Let $R$ be a commutative ring with identity. By a basis in $R$ we mean a finite set of elements which generate $R$. The order of a basis is the largest integer $n$ for which there exist $n+1$ members of the basis with nonzero product.

There is a close relation between bases in $C(X)$ and basic covers of $X$ [2, p. 243]. For each basis $\{f_i\}_{i \in I}$ of $C(X)$ we associate the basic cover $\{U_i\}_{i \in I}$ of $X$, where $U_i$ is defined by $U_i = \{x : f_i(x) \not= 0\}$. Conversely, for each basic cover $\{U_i\}_{i \in I}$ of $X$, we may associate a basis $\{f_i\}_{i \in I}$ of $C(X)$ where $f_i$ is chosen to satisfy $U_i = \{x : f_i(x) \not= 0\}$. Since $U_{i_1} \cap \cdots \cap U_{i_n} = \emptyset$ if and only if $f_{i_1} \cdots f_{i_n} = 0$, it follows that the order of the basic cover $\{U_i\}_{i \in I}$ is the same as the order of the basis $\{f_i\}_{i \in I}$.

If now $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are bases in the ring $R$, we say that $\{b_j\}_{j \in J}$ is a refinement of $\{a_i\}_{i \in I}$ if for each $j \in J$ there is an $i \in I$ such that $b_j$ is a multiple of $a_i$. The dimension of $R$, denoted by $d(R)$, is here defined to be the least cardinal $m$ such that every basis of $R$ has a refinement of order at most $m$.

Theorem. If $X$ is an arbitrary topological space, then $\dim X = d(C(X))$.

Proof. Suppose $d(C(X)) \leq n$. Let $\{U_i\}_{i \in I}$ be a basic cover of $X$ and let $\{f_i\}_{i \in I}$ be an associated basis in $C(X)$. By hypothesis, this basis has a refinement $\{g_j\}_{j \in J}$ of order at most $n$. The basic cover associated with $\{g_j\}_{j \in J}$ is then a refinement of $\{U_i\}_{i \in I}$ of order at most $n$. Thus $\dim X \leq n$.

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Suppose now that \( \text{dim } X \leq n \). Let \( \{f_i\}_{i \in I} \) be a basis in \( C(X) \) and let \( \{U_i\}_{i \in I} \) be the associated basic cover of \( X \). By hypothesis, this cover has a basic refinement \( \{V_j\}_{j \in J} \) of order at most \( n \). By Theorem 16.6 of [2], there are zero-sets \( Z_j \) which cover \( X \) and for which \( Z_j \subseteq V_j \) for \( j \in J \). Let \( k_j \in C(X) \) satisfy the following: \( k_j(x) = 0 \) if \( x \notin V_j \) and \( k_j(x) = 1 \) if \( x \in Z_j \). For each \( j \in J \), choose \( i(j) \in I \) such that \( V_j \subseteq U_{i(j)} \), and let \( g_j = k_j f_{i(j)} \). From the construction of \( g_j \) it follows that

\[
Z_j \subseteq \{x: g_j(x) \neq 0\} \subseteq V_j.
\]

Since the \( Z_j \)'s cover \( X \), it follows that \( \{g_j\}_{j \in J} \) is a basis of \( C(X) \), which clearly refines \( \{f_i\}_{i \in I} \). Since \( \{V_j\}_{j \in J} \) has order at most \( n \), the family of sets \( \{x: g_j(x) \neq 0\} \) also has order at most \( n \), whence the basis \( \{g_j\}_{j \in J} \) has order at most \( n \). Thus \( d(C(X)) \leq n \) and the proof is complete.

REFERENCES
