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AN EXACT SEQUENCE CALCULATION FOR THE SECOND HOMOTOPY OF A KNOT

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ABSTRACT. This paper deals with the computation of the second homotopy group of a knot, cancelling the action of the commutator subgroup of the fundamental group.

0. **Introduction and basic definitions.** A knot is a smooth embedding $k: S^n \to S^{n+2}$. Let $X = S^{n+2} - \mathrm{Im}(k)$. In [5] Kervaire characterizes $\pi_1(X)$ and describes $\pi_2(X)$ in the case $\pi_1(X) = \boldsymbol{Z}$.

In [3] Fox suggests that $\pi_2(X)$ should be described as a $\pi_1(X)$-module under the action described in [4, p. 125] and for the case of a spun knot, an answer has been found [8].

In this paper we follow a different approach: first, we consider the universal abelian cover $\tilde{X}$ of X, and observe that $\pi_2(X)$—which is isomorphic to $\pi_2(\tilde{X})$—is a group which $\boldsymbol{Z}$ acts on (as covering transformations) and we find a description for it, similar to that of Levine [7] for the case of homology groups.

It is known [6] that $k: S^n \to S^{n+2}$ always extends to an embedding $V^{n+1} \to S^{n+2}$ where V is a compact oriented manifold with boundary $\partial V = S^n$ called a Seifert manifold for k. Let Y be the $(n+2)$-sphere cut along V as in [6]; Y is a compact manifold (with a corner in $k(S^n)$) with boundary $\partial Y = V_0 \cup V_1$ where $V_t \approx V$ and the two copies of V are pasted along the boundary. The composition $V_t \approx V \subset Y$ gives a map ν_t. To establish some notation we set $\pi_1(V) = H$, $\pi_1(Y) = G$ and $\nu_t: H \to G$, the homomorphisms induced by the maps above. If V' (resp. $V'', \cdots$) is another Seifert manifold for k, we call Y', H', G', ν_t' (resp. $Y'', H'', \cdots$) the corresponding manifold, groups and homomorphisms. If both maps $\nu_t: H \to G$ are monomorphisms, we say that V is a minimal Seifert manifold.

Let Π be the fundamental group of X and $\tilde{\Pi}$ its commutator subgroup; there is an exact sequence

$$(1) \qquad 1 \to \tilde{\Pi} \to \Pi \to \boldsymbol{Z} \to 0.$$

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The covering $\tilde{X}$ associated to (1) can be constructed as in [**7**]: let $Y(n)$ be a copy of $Y-k(S^n)$ with boundary Int $V_0 \cup$ Int V_1; $\tilde{X}$ is obtained by taking the union of all the $Y(n)$ with identifications Int $V_0(n+1)=$ Int $V_1(n)$. This construction clearly describes the action of $\boldsymbol{Z}$ on $\tilde{X}$ as translations between the sheets $Y(n)$.

Let S be a space with fundamental group A and second homotopy B; by B_A we indicate the quotient of B by the action of A [**4**, p. 125]. If S is a space where $\boldsymbol{Z}$ acts, its homotopy is naturally a module over the integral group ring of $\boldsymbol{Z}$, henceforth called Λ and viewed as the ring of Laurent polynomials with integral coefficients $\boldsymbol{Z}[t, t^{-1}]$.

The purpose of this note is to establish the following result:

THEOREM 0. *Let k be a knot ($n \geqq 3$) and V a minimal Seifert manifold for it; the following is an exact sequence of Λ-modules*:

$$\pi_2(V)_H \otimes_{\boldsymbol{Z}} \Lambda \xrightarrow{d} \pi_2(Y)_G \otimes_{\boldsymbol{Z}} \Lambda \longrightarrow \pi_2(X)_{\tilde{\Pi}} \longrightarrow 0 \tag{2}$$

where d is given by $\alpha \otimes 1 \to v_{0*}\alpha \otimes t - v_{1*}\alpha \otimes 1$ with v_{t*} being the homomorphisms of second homotopy induced by the maps $v_t: V \to Y$.

Sequence (2) is trivial for $n=1$ and always exact for $n \geqq 3$; for $n=2$ it is difficult to know whether minimal Seifert manifolds exist. For spun knots such a manifold is easy to obtain and (2) is a weaker restatement of [**8**]. Finally for some knots constructed by Sumners, an obvious (minimal) Seifert manifold exists and (2) gives a new proof of Theorem 1 of [**12**].

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1. **The fundamental group.** With the notation of §0, recall that if $\alpha \in H$ is in ker v_t, for some t, the Seifert manifold V can be exchanged for a new one $V' \approx \chi(V, \beta)$ where $\beta: S^1 \times D^n \to$ Int V is a map representing α; see [**6**, p. 12].

If r is an integer, let $F(r)$ be the free group on r generators. We know [**10**] that both H and G are finitely generated groups, so there is an epimorphism $\phi: F(r) \to G$ for some r. For $\alpha \in G$, the Seifert manifold $V \#_t \alpha$ ($t=0, 1$) is obtained in the following way: represent α by a differentiable map $f: S^1 \to Y$ which can be extended to $g: S^1 \times S^n \to Y$ whose image is the boundary of a tubular neighborhood of Im(f). Let $\gamma: I \to Y$ be an arc entirely contained in Y and with endpoints $\gamma(0) \in V_t$ and $\gamma(1) \in$ Im(g). The manifold $V \#_t \alpha$ is simply $V_t \#$ Im(g) along γ, considered as a submanifold of S^{n+2}.

LEMMA 1. *If k is a knot of dimension $n \geqq 3$, there exists a minimal Seifert manifold for k.*

PROOF. Start with V being any Seifert manifold and let $x_1, \cdots, x_r$ be generators for G; let $V'=V \#_0 x_1 \#_0 \cdots \#_0 x_r$. By the Van Kampen theorem, $H'=H*F(r)$ and $G'=G*F(r)$ where $*$ indicates free product, and

$$\nu_0' \,|\, H = \nu_0, \qquad \nu_1' \,|\, H = \nu_1,$$
$$\nu_0' \,|\, F(r) = \phi, \qquad \nu_1' \,|\, F(r) = 1_{F(r)}.$$

Thus, ν_0' maps H' onto G, a finitely presented group and so, by [**1**, Lemma IV.1.15], ker ν_0' has finite weight and by a remark above, V' can be replaced by a new manifold V'' with ker $\nu_0''=0$. By the same reasoning we can assume that ker ν_1'' has finite weight. Let $\xi \in$ ker ν_1'' and V''' be the Seifert manifold obtained by adding a handle h^2 to V'' along ξ; suppose $\alpha \in$ ker ν'''; then $\alpha : S^1 \to V'''$ can be deformed to miss h^2. Naturally, α extends to $\beta : D^2 \to Y'''$ since $\alpha \in$ ker ν'''_0, and $\beta(D^2)$ must hit V'' else α would be in ker ν_0''. It is clear that the intersection lies in a tubular neighborhood of Im(ξ) used to attach h^2 and on the "zero" side of V''. Thus β describes a homotopy in Y'' of $\nu_0''(\alpha)$ with $\nu_0''(c\xi)$ for some $c \in \boldsymbol{Z}$. Since ν_0'' is a monomorphism $\alpha = c\xi$. This means that α is nullhomotopic in V'''. In that manner we know that ker ν_1'' can be eliminated by ambient surgery to obtain V^* without disturbing ker ν_0'' in the process.

We can now describe $\tilde{\Pi}$ as in [**9**, Theorem 4.5.1.]:

$$\tilde{\Pi} = \cdots *_H G *_H G *_H G *_H \cdots \tag{3}$$

where $G*_H G$ indicates the free product with amalgamation as defined in [**15**, p. 231]. The amalgamating maps are the ν_t which—by Lemma 1—can be taken to be monomorphisms.

2. **Some exact sequences.** From (3) we see that $\boldsymbol{Z}$ acts naturally on $\tilde{\Pi}$ and by the Mayer-Vietoris sequence for homology of groups with trivial $\boldsymbol{Z}$-coefficients [**11**], we obtain a long exact sequence of Λ-modules:

$$\begin{aligned} \cdots \longrightarrow H_q(H) \otimes_{\boldsymbol{Z}} \Lambda \xrightarrow{d} H_q(G) \otimes_{\boldsymbol{Z}} \Lambda \\ \longrightarrow H_q(\tilde{\Pi}) \longrightarrow H_{q-1}(H) \otimes_{\boldsymbol{Z}} \Lambda \longrightarrow \cdots \end{aligned} \tag{4}$$

where d is given by the map $\alpha \otimes 1 \to \nu_0 \alpha \otimes t - \nu_1 \alpha \otimes 1$. (See also [**7**].)

We use Trotter's techniques to prove:

LEMMA 2. *The map $i_* : H_3(G) \otimes \Lambda \to H_3(\tilde{\Pi})$ is onto.*

PROOF. Suppose the group G is presented as $\langle y_1, \cdots, y_\alpha : R_1, \cdots, R_\beta \rangle$; call $G^{(n)}$ ($n \in \boldsymbol{Z}$) the copies of G occurring in (3), with elements $g^{(n)}$. Let

$\nu_t^{(n)}: H \to G^{(n)}$ be the map ν_t. In that case, $\tilde{\Pi}$ has a presentation $\langle y_i^{(n)}: R_j^{(n)}, \nu_0^{(n+1)} = \nu_1^{(n)} \rangle$ where $i=1, \cdots, \alpha$; $j=1, \cdots, \beta$ and the expression $\nu_0^{(n+1)} = \nu_1^{(n)}$ represents the relations $\nu_0^{(n+1)}(h) = \nu_1^{(n)}(h)$ for all $h \in H$, $n \in \boldsymbol{Z}$. Finally, by (1), Π is presented as $\langle y_i, t: R_j, t\nu_0 t^{-1} = \nu_1 \rangle$.

Consider the subgroup J of the $\alpha \in H$ such that $\nu_0 \alpha = \nu_1 \alpha$; it is a normal subgroup and say $\langle A_1, \cdots, A_\gamma \rangle = J$ where γ is not necessarily finite. Consider symbols $x_1, \cdots, x_\gamma$ and define $r_i = [t, x_i]$ and $s_i = x_i^{-1}(\nu_0 A_i)$.

We then have

$$\Pi = \langle t, x_1, \cdots, x_\gamma, y_1, \cdots, y_\alpha : R_1, \cdots, R_\beta, t\nu_0 t^{-1} = \nu_1, r_1, \cdots, r_\gamma, s_1, \cdots, s_\gamma \rangle. \tag{5}$$

To obtain a complete set of identities for Π, it is necessary to find expressions for the r_i as consequence of other relations; this is done as in [**14**, pp. 481–482] and yields identities ι_γ, one for each A_γ, expressed by formula (4.1) of [**14**], and the identities c_k of the group G that arise from the presence of the relators R_j in (5).

To calculate $H_*(G)$ we use the complex

$$Y_3 \otimes_G \boldsymbol{Z} \to Y_2 \otimes_G \boldsymbol{Z} \to \cdots$$

where Y_3 is the free G-module in the c_k; similarly, the complex

$$X_3 \otimes_\Pi \Lambda \xrightarrow{d_3} X_2 \otimes_\Pi \Lambda \longrightarrow \cdots,$$

with X_3 the free Π-module on the ι_γ and c_k, is used to compute $H_*(\tilde{\Pi})$. (See [**2**, p. 196].)

Thus X_3 is the direct sum of two Π-submodules $X_3 = I \oplus C$ where I is generated by $\{\iota_\gamma\}$ and C, by $\{c_k\}$.

X_2 is in turn the free Π-module generated by symbols representing the relations in presentation (5), say R_j, ρ_l, r_γ, s_γ, where ρ_l represents one relation of the form $t\nu_0 t^{-1} = \nu_1$. As in [**14**, Proposition 4.1], $d_3(\iota_\gamma \otimes 1) = r_\gamma - (1-t)s_\gamma$, which implies that d_3 is a monomorphism when restricted to $I \otimes \Lambda$.

As a result, if $Z \in \ker d_3$, $Z \in C \otimes \Lambda$, which is the image of the map $i: (Y_3 \otimes_G \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \Lambda \to X_3 \otimes_\Pi \Lambda$. This map induces the inclusion $H_3(G) \otimes \Lambda \to H_3(\tilde{\Pi})$. Thus if $Z \in H_3(\tilde{\Pi})$, Z can be represented as an element in $C \otimes \Lambda = \operatorname{Im}(i)$ and the lemma follows.

Let $\tilde{X}$ be the covering of G associated to (1). In [**7**], the following exact sequence is found:

$$\cdots \longrightarrow H_q(V) \otimes_{\boldsymbol{Z}} \Lambda \xrightarrow{d} H_q(Y) \otimes_{\boldsymbol{Z}} \Lambda \longrightarrow H_q(\tilde{X}) \longrightarrow H_{q-1}(V) \otimes_{\boldsymbol{Z}} \Lambda \longrightarrow \cdots \tag{6}$$

where d is defined as in (4). Let $W = S^{n+2} - Y$; then W is a deformation retraction of V and $W \cap Y = V_0 \cup V_1$. From the Mayer-Vietoris sequence for $Y \cup W = S^{n+2}$, we obtain:

$$(7) \qquad 0 \longrightarrow H_q(V) \oplus H_q(V) \xrightarrow{e} H_q(Y) \oplus H_q(V) \longrightarrow 0$$

where $e(\alpha, \beta) = (\nu_0\alpha - \nu_1\beta, \alpha - \beta)$. Thus $e\Delta$, where $\Delta: H_q(V) \to H_q(V) \oplus H_q(V)$ is the diagonal map, is a monomorphism $H_q(V) \to H_q(Y)$ given by $\alpha \to \nu_0\alpha - \nu_1\alpha$. Since Λ is noetherian, d is an isomorphism. (I am indebted to D. W. Sumners for pointing out to me this last step.)

Therefore (6) splits into short exact sequences

$$(6.q) \qquad 0 \longrightarrow H_q(V) \otimes_Z \Lambda \xrightarrow{d} H_q(Y) \otimes_Z \Lambda \longrightarrow H_q(\tilde{X}) \longrightarrow 0.$$

Lemma 2 and (6.1) indicate that

$$(4.q) \qquad 0 \longrightarrow H_q(H) \otimes_Z \Lambda \xrightarrow{d} H_q(G) \otimes_Z \Lambda \longrightarrow H_q(\tilde{\Pi}) \longrightarrow 0$$

are exact for $q = 1, 2$ and right exact for $q = 3$.

Recall [2] that by a theorem of Hopf we have sequences

$$(8) \qquad H_3(\tilde{X}) \to H_3(\tilde{\Pi}) \to (\pi_2(X))_{\tilde{\Pi}} \to H_2(\tilde{X}) \to H_2(\tilde{\Pi}) \to 0,$$

$$(9) \qquad H_3(Y) \to H_3(G) \to (\pi_2(Y))_G \to H_2(Y) \to H_2(G) \to 0,$$

$$(10) \qquad H_3(V) \to H_3(H) \to (\pi_2(V))_H \to H_2(V) \to H_2(H) \to 0.$$

Since Λ is free, we can tensor (8), (9) and (10) with Λ and put sequences (4.q), (6.q), (8), (9) and (10) together to get

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
H_3(\tilde{X}) & \to & H_3(\tilde{\Pi}) & \to & (\pi_2(X))_{\tilde{\Pi}} & \to & H_2(\tilde{X}) & \to & H_2(\tilde{\Pi}) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
H_3(Y) \otimes_Z \Lambda & \to & H_3(G) \otimes_Z \Lambda & \to & (\pi_2(Y))_G \otimes_Z \Lambda & \to & H_2(Y) \otimes_Z \Lambda & \to & H_2(G) \otimes_Z \Lambda & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
H_3(V) \otimes_Z \Lambda & \to & H_3(H) \otimes_Z \Lambda & \to & (\pi_2(V))_H \otimes_Z \Lambda & \to & H_2(V) \otimes_Z \Lambda & \to & H_2(H) \otimes_Z \Lambda & \to & 0 \\
\uparrow & & & & & & \uparrow & & \uparrow & & \\
0 & & & & & & 0 & & 0 & &
\end{array}$$

A simple application of the five lemma makes the middle vertical sequence an exact sequence of Λ-modules. This proves Theorem 0.

3. **Applications.** From [12, p. 356] we conclude that the sequence

$$H_3(X) \to H_3(\Pi) \to (\pi_2(X))_\Pi \to H_2(X)$$

is exact. By Alexander duality $H_3(\Pi) \approx (\pi_2(X))_\Pi$. Thus, sequence (2) yields the following result:

LEMMA 3. *The following is an exact sequence of groups*:

$$\pi_2(V)_H \xrightarrow{d} \pi_2(Y)_G \longrightarrow H_3(\Pi) \longrightarrow 0 \tag{11}$$

where d is given by $\alpha \to \nu_0\alpha - \nu_1\alpha$.

This is particularly interesting in the case of fibered knots $k: S^n \to S^{n+2}$ with fibre F with fundamental group D. In that case (2) is exact for $n=2$ because F is a minimal Seifert manifold. Let $A = \pi_2(F)_D$; since F is a 3-manifold, A is a finitely generated abelian group. Thus $H_3(\Pi)$ is always finitely generated in view of (11).

If D is finite of order d, $A = \mathbf{Z}_d$ [**13**, Theorem 12] and (11) becomes

$$\mathbf{Z}_d \xrightarrow{h_* - I} \mathbf{Z}_d \longrightarrow H_3(\Pi) \longrightarrow 0$$

where $h: F \to F$ is the pasting homomorphism. By [**13**, Theorem 13], $h_* = I$ which means $H_3(\Pi) = \mathbf{Z}_d$; furthermore $H_3\tilde{\Pi}$ is a trivial Λ-module because the map d in (4.3) is $t-1$ and so $H_3\Pi = H_3\tilde{\Pi}$.

PROPOSITION 4. *Let* $k: S^2 \to S^4$ *be a fibered knot and* $\Pi = \pi_1(S^4 - k(S^2))$; *then* $H_3(\Pi)$ *is finitely generated and, if the fibre has finite fundamental group of order d*,

$$H_3\Pi = H_3\tilde{\Pi} = \mathbf{Z}_d.$$

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