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GENERALIZED RAMSEY THEORY FOR GRAPHS. II. SMALL DIAGONAL NUMBERS

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ABSTRACT. Consider a finite nonnull graph G with no loops or multiple edges and no isolated points. Its *Ramsey number* $r(G)$ is defined as the minimum number p such that every 2-coloring of the lines of the complete graph K_p must contain a monochromatic G. This generalizes the classical diagonal Ramsey numbers $r(n, n) = r(K_n)$. We obtain the exact value of the Ramsey number of every such graph with at most four points.

1. **A celebrated Putnam question.** The following question (see [3]) was already well known to most of those who knew it. Independently, it found its way into a Putnam examination where it attracted much attention:

"Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other."

Stated in the natural language [5] of graph theory, this asserts that whenever each of the 15 lines of the complete graph K_6 is colored either green or red, there is at least one monochromatic triangle.

Actually, there are at least two such triangles, as proved by Goodman [3]. Since we cannot color the lines of a graph green and red, we use solid and dashed lines instead in all the figures.

We proposed in [1] the more general approach of 2-coloring the lines of any graph G and investigating whether there must occur a monochromatic copy of a specified subgraph F. Henceforth, a *2-coloring* of G will mean a coloration of the lines of G with the two colors green and red.

A simple example (Figure 1) illustrating this viewpoint is obtained when we set $G = C_5$ and $F = P_3$. Whenever one colors the five lines of C_5 with two colors, there must obviously occur a monochromatic P_3.

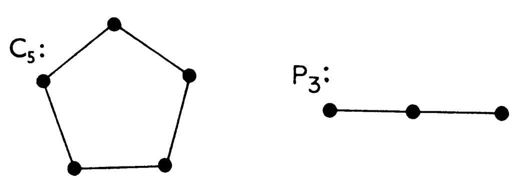

FIGURE 1.

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2. **The diagonal Ramsey numbers.** The diagonal Ramsey number $r(n, n)$ is defined [**5**, p. 16] as the smallest p such that in any 2-coloring of the complete graph K_p, there always occurs a monochromatic K_n.

Generalizing this concept, we now define *the Ramsey number* $r(F)$ *for any graph F with no isolated points*. The value of $r(F)$ is the smallest p such that in every 2-coloring of K_p, there always occurs a monochromatic F. (This definition of $r(F)$ coincides with that of $r(F, 2)$ introduced in [**2**].) In particular, we have $r(n, n) = r(K_n)$, and trivially $r(K_2) = 2$. The Putnam problem mentioned above amounts to showing that $r(K_3) \leqq 6$. In fact, $r(K_3) = 6$ because the ten lines of K_5 can be colored green and red in such a manner that no monochromatic K_3 occurs. There is only one such 2-coloring (Figure 2), namely that which gives rise to a red C_5 and a green C_5 (pentagon and pentagram).

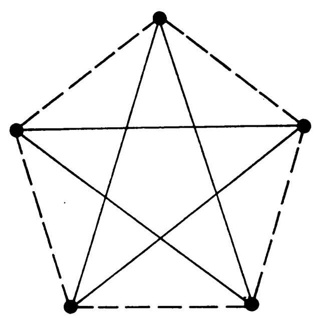

FIGURE 2.

Greenwood and Gleason [**4**] proved that $r(K_4) = 18$ by (a) producing a 2-coloring of K_{17} which has no monochromatic K_4, and (b) showing elegantly that every 2-coloring of K_{18} does contain such a K_4. Although upper and lower estimations for $r(K_n)$ are known, the exact values of $r(K_n)$ with $n \geqq 5$ are still entirely open. Thus the determination of $r(F)$ for the graphs with at most four points would bring us just up to $r(K_5)$. It is our object to calculate $r(F)$ exactly for these small graphs.

3. **All stars.** The Ramsey numbers of the stars are

$$
\begin{aligned}
(1) \qquad r(K_{1,m}) &= 2m, && m \text{ odd}, \\
&= 2m - 1, && m \text{ even}.
\end{aligned}
$$

We first prove (1) for odd m. In this case, there is a regular graph G of degree $m-1$ having $2m-1$ points, so its complement $\bar{G}$ is regular of degree $m-1$. Hence the decomposition (2-coloring) of K_{2m-1} into G and $\bar{G}$ shows that $r(K_{1,m}) \geqq 2m$. The equality holds for in any 2-coloring of K_{2m}, the green and red degrees of each point u sum to $2m-1$, whence one of these degrees is at least m.

When m is even, if there is a 2-coloring of K_{2m-1} without a monochromatic star $K_{1,m}$, then both the green and red degree of each point equal $m-1$. But then the green graph is regular of degree $m-1$, which is a contradiction as both $m-1$ and $2m-1$ are odd. Thus we have $r(K_{1,m}) \leqq 2m-1$. The equality follows from a decomposition of K_{2m-2} into G and $\bar{G}$, where G is a regular graph of degree $m-1$ with $2m-2$ points.

4. **Small generalized Ramsey numbers.** There are exactly ten graphs F (Figure 3) with at most 4 points, having no isolates. We now find $r(F)$ for

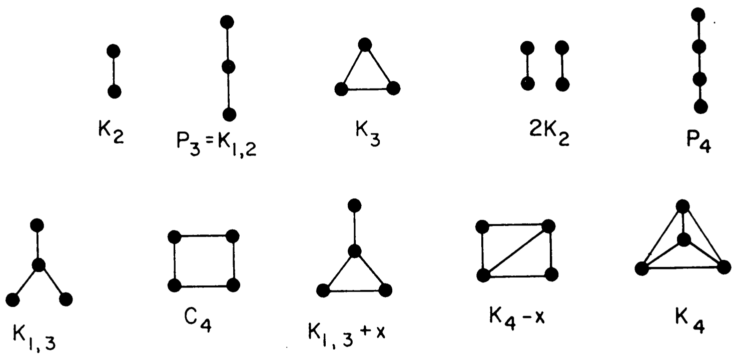

FIGURE 3.

each of these. For convenience in identifying them, we use the operations on graphs from [5, p. 21], to get a symbolic name for each.

We have already seen that $r(K_2)=2$, $r(K_3)=6$ and $r(K_4)=18$. Setting $m=2$ and $m=3$ in (1), we obtain $r(K_{1,2})=3$ and $r(K_{1,3})=6$. Thus there are just five more graphs to investigate: $2K_2$, P_4, C_4, $K_{1,3}+x$ and K_4-x.

$r(2K_2)=5$. There is a 2-coloring of K_4 (Figure 4) with no monochromatic $2K_2$. On the other hand, it is ridiculously simple to verify that there is no such 2-coloring of the cycle C_5, *a fortiori* of K_5.

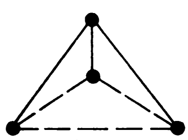

FIGURE 4.

$r(P_4)=5$. By coincidence, Figure 4 shows that $r(P_4)>4$. We now exploit the fact, just noted, that every 2-coloring of K_5 has a monochromatic $2K_2$. Let u_1u_2 and v_1v_2 be two independent green lines in K_5. While trying to avoid a green P_4, we must color all four lines u_iv_j red, thus producing an all red P_4, namely $u_1v_1u_2v_2$.

$r(C_4)=6$. Luckily, Figure 2 shows that $r(C_4)>5$.

Now assume there is a 2-coloring of K_6 with no monochromatic 4-cycle, C_4. As we already have $r(K_3)=6$, there is a (say) green triangle $u_1u_2u_3$ in K_6. Let v_1, v_2, v_3 be the other points. From each v_i, there is at most one green line to this green triangle, for otherwise, we have a green C_4. We now show that from each v_i, there is *exactly* one green line to the triangle. If not, all three lines u_iv_1 are red. But then the fact that at least two lines u_iv_2 are red gives a red C_4, like $v_1u_2v_2u_3v_1$. Next we rule out the possibility that there is more than one green line from any u_i to the v_j, as shown in Figure 5(a) for u_2. This is seen from the red lines in Figure 5(b) which are forced while trying to avoid a green C_4.

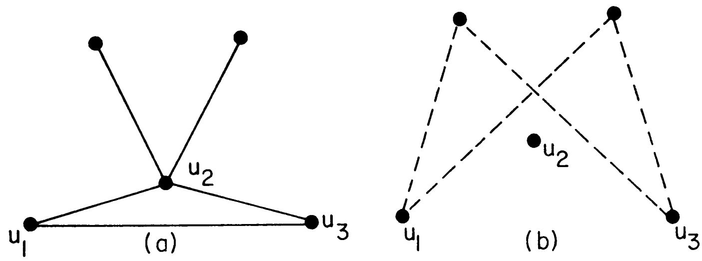

FIGURE 5.

Now we know that there are green lines in this K_6 which must look like Figure 6, with no other green u_iv_j lines.

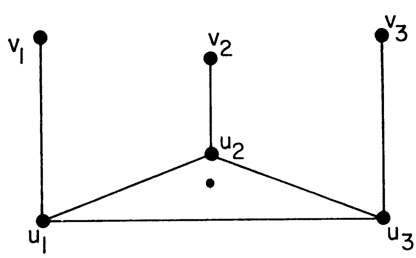

FIGURE 6.

Clearly all the lines v_iv_j are red. And now we have got it, because $v_1v_2v_3u_2v_1$ is a red C_4.

$r(K_{1,3}+x)=7$. The 2-coloring of K_6 in which $2K_3$ is red and $K_{3,3}$ is green (Figure 7) shows that $r(K_{1,3}+x)>6$. To prove that $r(K_{1,3}+x)=7$, we will show that it is impossible to have a 2-coloring of K_7 without a monochromatic $K_{1,3}+x$. To begin, we know by $r(K_3)=6$ that K_7 has (say) a green K_3 with points u_1, u_2, u_3. Call the other points v_1 to v_4. To avoid an immediate green $K_{1,3}+x$, we need to color all 12 lines u_iv_j red (obtaining a

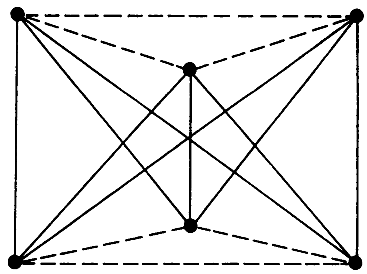

FIGURE 7.

red $K_{3,4}$). Next to avoid a sudden red $K_{1,3}+x$, all 6 of the lines v_iv_j must be green. But behold we have a green K_4, hence *a fortiori* a green $K_{1,3}+x$.

$r(K_4-x)=10$. If one stumbles on the correct example quickly (we did not), it is not at all difficult to see that $r(K_4-x)>9$. This example, which we believe to be the unique correct 2-coloring of K_9, is given by taking the cartesian product $K_3\times K_3$ of two triangles as the green subgraph. Figure 8 shows only the green lines; those which are absent are red. Clearly, neither $K_3\times K_3$ nor its complement contains K_4-x.

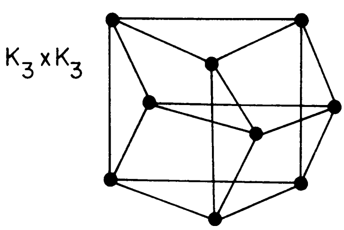

FIGURE 8.

We now prove that $r(K_4-x)=10$. Consider an arbitrary 2-coloring of K_{10}. By (1), there is a monochromatic (say green) $K_{1,5}$, or in other words a point u adjacent greenly to 5 points u_i, $i=1$ to 5. We can now ignore the other four points and concentrate on the 10 lines u_iu_j. There are two possibilities. If there is a green P_3 on the points u_i, say $u_1u_2u_3$, then these 2 lines together with the 3 lines $u\,u_j$, $j=1, 2, 3$, form a green K_4-x. On the other hand, if there is no green P_3 on the u_i, then there are at most two green lines u_iu_j. But every red graph with 5 points and 8 lines must contain a red K_4-x, completing the proof.

5. **Conclusions.** The small generalized diagonal Ramsey numbers just established are summarized in the following table:

F	K_2	P_3	K_3	$2K_2$	P_4	$K_{1,3}$	C_4	$K_{1,3}+x$	K_4-x	K_4
$r(F)$	2	3	6	5	5	6	6	7	10	18

The next paper [2] in this series derives exact values of the small generalized off-diagonal Ramsey numbers for the above graphs F. These are defined on pairs of graphs F_1, F_2 as the smallest p such that any 2-coloring of K_p contains either a green F_1 or a red F_2. In another sequel [6], all the explicit 2-colorings of K_6 with the minimum number (two) of monochromatic triangles are displayed.

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