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DIRICHLET FINITE BIHARMONIC FUNCTIONS ON THE UNIT DISK WITH DISTORTED METRICS[1]

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ABSTRACT. The Riemannian manifold $\boldsymbol{D}_\alpha$ obtained from the unit disk $\boldsymbol{D}$ by giving the distorted metric $(1-|z|)^{-\alpha}|dz|$ does not admit any Dirichlet finite nonharmonic biharmonic function if and only if $\alpha \geqq \frac{3}{4}$.

1. The Laplace-Beltrami operator Δ on a smooth manifold M with a smooth Riemannian metric $ds^2=\sum_{ij} g_{ij}(x)\, dx^i\, dx^j$ applied to a smooth function φ takes the form $\Delta\varphi=g^{-1/2}\sum_{ij}(g^{1/2}g^{ij}\varphi_{x^i})_{x^j}$. Functions in the class $H^2(M)=\{u\in C^4(M);\ \Delta^2 u=0\}$ are called biharmonic. Let $D(M)$ be the class of functions φ on M having square integrable gradients, i.e. the Dirichlet integrals $D_M(\varphi)=\int_M |\text{grad}\ \varphi|^2 * 1$ are finite. Consider the class $\mathcal{O}_{H^2D}$ of manifolds M such that $H^2D(M)=HD(M)$, where $H(M)=\{u\in C^2(M);\ \Delta u=0\}$ is the class of harmonic functions on M. Thus $M\in\mathcal{O}_{H^2D}$ if and only if there exists no Dirichlet finite nonharmonic biharmonic function on M.

In a recent paper [**2**], Nakai and Sario made a detailed study on H^2D and $\mathcal{O}_{H^2D}$, among which we are concerned with the following result. Let $\boldsymbol{D}$ be the unit disk $|z|<1$ and $\boldsymbol{D}_\alpha$ be the disk $\boldsymbol{D}$ equipped with the Riemannian metric $ds=(1-|z|)^{-\alpha}|dz|$. They showed that $\boldsymbol{D}_\alpha\notin\mathcal{O}_{H^2D}$ $(\alpha<\frac{3}{4})$ and $\boldsymbol{D}_\alpha\in\mathcal{O}_{H^2D}$ $(\alpha>\frac{3}{4})$, but the case for $\alpha=\frac{3}{4}$ is left unsettled. The purpose of the present note is to report that this gap can be filled by showing $\boldsymbol{D}_{3/4}\in\mathcal{O}_{H^2D}$. Therefore we obtain

THEOREM. *The manifold $\boldsymbol{D}_\alpha$ belongs to the null class $\mathcal{O}_{H^2D}$ if and only if* $\alpha\geqq\frac{3}{4}$.

It is interesting to compare this with the following result in a recent paper of Nakai [**1**]. Let a nonnegative smooth $P(z)\sim(1-|z|)^{-2\alpha}$ $(|z|\to 1)$. The equation $\Delta u=Pu$ has no Dirichlet finite solution on $\boldsymbol{D}$ if and only if $\alpha\geqq\frac{3}{4}$. As in [**1**] even if we replace $(1-|z|)^{-\alpha}$ by a smooth rotation free function $\lambda_\alpha(z)$ in the sense that $\lambda_\alpha(z)\sim(1-|z|)^{-\alpha}$ $(|z|\to 1)$ in the definition

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of $\boldsymbol{D}_\alpha$, still we can prove the validity of the above theorem. How to modify the proof for $(1-|z|)^{-\alpha}$ so that it also serves for $\lambda_\alpha(z)$ should be clear.

Before proceeding to the proof for $\boldsymbol{D}_{3/4}\in\mathcal{O}_{H^2D}$ the author must mention his indebtedness to M. Nakai and L. Sario for showing him the preprints of the papers and suggesting this problem to him. Except for the technical precisions, the main idea of the author's proof is due to Nakai and Sario **[2]**.

2. To prove $\boldsymbol{D}_{3/4}\in\mathcal{O}_{H^2D}$ we only have to show that the Poisson equation $\Delta u=-h$ on $\boldsymbol{D}_{3/4}$ has no Dirichlet finite solution for every choice of $h\in H(\boldsymbol{D}_{3/4})$. Here we must remark that $H(\boldsymbol{D}_{3/4})=H(\boldsymbol{D})$, $D_{\boldsymbol{D}_{3/4}}(\varphi)=D_{\boldsymbol{D}}(\varphi)$, and $D(\boldsymbol{D}_{3/4})=D(\boldsymbol{D})$, where $\boldsymbol{D}$ is understood to be $\boldsymbol{D}_0$.

Fix an arbitrary $h\in H(\boldsymbol{D})$, $h\neq 0$, and expand it into its Fourier series:

$$h(re^{i\theta}) = \sum_{\nu=0}^{\infty} r^\nu(a_\nu \cos \nu\theta + b_\nu \sin \nu\theta). \tag{1}$$

Since $h\not\equiv 0$, there exists at least one ν such that $a_\nu^2+b_\nu^2\neq 0$, which we fix throughout our proof. Let $t\in(\frac{1}{2}, 1)$ and a function $\rho_t(r)$ defined for $r\in[0, 1)$ be given by

$$\begin{aligned} \rho_t(r) &= 2r, & r &\in [0, \tfrac{1}{2}); \\ &= 1, & r &\in [\tfrac{1}{2}, t); \\ &= 1 - 2(1-t)^{-1}(r-t), & r &\in [t, (t+1)/2); \\ &= 0, & r &\in [(t+1)/2, 1). \end{aligned} \tag{2}$$

We then consider a function $\varphi_t\in C_0D(\boldsymbol{D})$ defined by

$$\varphi_t(re^{i\theta}) = \rho_t(r)(a_\nu \cos \nu\theta + b_\nu \sin \nu\theta), \tag{3}$$

where $C_0(\boldsymbol{D})$ is the class of continuous functions with compact supports in $\boldsymbol{D}$.

3. We denote by dv the Riemannian volume element $(1-r)^{-3/2}r\,dr\,d\theta$ on $\boldsymbol{D}_{3/4}$. If τ is a positive number, then by an easy computation we obtain the estimate

$$\int_{\boldsymbol{D}_{3/4}} h\cdot\tau\varphi_t\,dv \geqq A(\tau(1-t)^{-1/2}-\tau), \tag{4}$$

where A is a finite positive constant independent of t and τ. We denote by $\boldsymbol{D}'$ the open set $\frac{1}{2}<|z|<1$. Again by an elementary computation we find a finite positive constant B independent of t and τ such that

$$D_{\boldsymbol{D}'}(\tau\varphi_t) \leqq B(\tau^2(1-t)^{-1}+\tau^2). \tag{5}$$

4. Let $\{t_k\}_1^\infty$ be a sequence of real numbers in $(\frac{1}{2}, 1)$ such that $t_{k+1} > (t_k+1)/2$ $(k=1, 2, \cdots)$. Clearly $1-t_k < 2^{-k}$ $(k=1, 2, \cdots)$. Next we consider a sequence $\{\tau_k\}_1^\infty$ given by

$$\tau_k = (1 - t_k)^{1/2} k^{-1} \qquad (k = 1, 2, \cdots). \tag{6}$$

Observe that

$$\sum_{k=1}^{\infty} \tau_k < \infty, \qquad \sum_{k=1}^{\infty} \tau_k^2 < \infty. \tag{7}$$

We then consider a sequence $\{\Phi_n\}_1^\infty$ of functions Φ_n in $C_0D(\boldsymbol{D})$ given by

$$\Phi_n(re^{i\theta}) = \sum_{k=1}^{n} \tau_k \varphi_{t_k}(re^{i\theta}). \tag{8}$$

By (4) and (6) we deduce that

$$\int_{D_{3/4}} h\Phi_n \, dv \geqq A\left(\sum_{k=1}^{n} k^{-1} - \sum_{k=1}^{n} \tau_k\right). \tag{9}$$

By the definition of Φ_n, we see that $(\partial\Phi_n/\partial x^i)^2 = \sum_{k=1}^n (\tau_k\, \partial\varphi_k/\partial x^i)^2$ $(i=1, 2)$ except at points $|z| = t_1, \cdots, t_n$, and a fortiori $D_{D'}(\Phi_n) = \sum_{k=1}^n D_{D'}(\tau_k\varphi_k)$. By (5) we obtain

$$D_{D'}(\Phi_n) \leqq B\left(\sum_{k=1}^{n} k^{-2} + \sum_{k=1}^{n} \tau_k^2\right).$$

We denote by $\boldsymbol{D}''$ the open set $|z| < \frac{1}{2}$ and put $\psi = \varphi_t | \boldsymbol{D}''$ which is independent of t. Set $C = D_{D''}(\psi)$. Then since $\Phi_n | \boldsymbol{D}'' = \sum_{k=1}^n \tau_k \psi$, we have

$$D_{D''}(\Phi_n) \leqq \left(\sum_{k=1}^{n} \tau_k\right)^2 C.$$

Since $D_{D_{3/4}}(\Phi_n) = D_D(\Phi_n) = D_{D'}(\Phi_n) + D_{D''}(\Phi_n)$, we conclude that

$$D_{D_{3/4}}(\Phi_n) \leqq B \cdot \left(\sum_{k=1}^{n} k^{-2} + \sum_{k=1}^{n} \tau_k^2\right) + C \cdot \left(\sum_{k=1}^{n} \tau_k\right)^2. \tag{10}$$

From (9), (10), and (7), it follows that

$$\lim_{n\to\infty} \left(\int_{D_{3/4}} h\Phi_n \, dv\right)^2 \Big/ D_{D_{3/4}}(\Phi_n) = \infty. \tag{11}$$

5. We can now conclude that the Poisson equation $\Delta u = -h$ has no Dirichlet finite solution on $\boldsymbol{D}_{3/4}$. Contrary to our assertion suppose there existed a Dirichlet finite solution u on $\boldsymbol{D}_{3/4}$. By the Stokes theorem we deduce

$$\int_{D_{3/4}} h\Phi_n \, dv = D_{D_{3/4}}(u, \Phi_n).$$

By the Schwarz inequality

$$\left(\int_{D_{3/4}} h\Phi_n \, dv\right)^2 \Big/ D_{D_{3/4}}(\Phi_n) \leqq D_{D_{3/4}}(u). \tag{12}$$

This contradicts (11), and the proof for $D_{3/4} \in \mathcal{O}_{H^2D}$ is herewith complete.

References

1. M. Nakai, *The equation* $\Delta u = Pu$ *on the unit disk with almost rotation free* $P \geqq 0$, J. Differential Equations **10** (1972).

2. M. Nakai and L. Sario, *Existence of Dirichlet finite biharmonic functions* (to appear).

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