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SETS OF MULTIPLICITY AND DIFFERENTIABLE FUNCTIONS

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ABSTRACT. The paper contains two theorems relating the fine structure of differentiable functions, in one or more dimensions, to the behavior of Fourier-Stieltjes transforms on sets that are small in various ways.

In this paper we prove two theorems on the transformation of certain sets, defined as follows. A set E in a metric space is an *L-set* if there are sequences $\varepsilon_k \to 0$ and $\delta_k \to 0$, and for each k a decomposition $E = \bigcup_I E_i$, wherein $\operatorname{diam}(E_i) \leqq \varepsilon_k \delta_k$, while $d(E_i, E_{i'}) \geqq \delta_k$ $(i \neq i')$. For each compact L-set E of real numbers there is a function h of class $C^1(-\infty, \infty)$ with $h' > 0$, so that $h(E)$ is a Kronecker set ([2], [3]). The first theorem is a complement to this.

THEOREM I. *Let ω be a monotone, positive function on $(0, \infty)$, and $\omega(0+) = 0$; let C^1_ω be the set of functions φ in C^1 with $\varphi' > 0$, $|\varphi'(a) - \varphi'(b)| \leqq \omega(|a-b|)$ for all real a and b. Then there is a compact L-set E so that $\varphi(E)$ is an M_0-set for each φ in C^1_ω.*

To prove the theorem we choose a sequence of positive numbers (c_n) so that $c_0 = 1$, $\omega(c_n) < n^{-2}$, and $c_{n+1} < n^{-3} c_n$. We now construct finite sets F_n and E_n; the peculiar construction of F_n is the main point in the argument. F_n is a sequence of n^2 elements

$$x(m) = x(0) + mc_n + m^2 c_n n^{-5/2}, \qquad 1 \leqq m \leqq n^2.$$

Here $x(0) = -c_n - c_n n^{-5/2}$ so that $x(1) = 0$. E_n is then a union of translates of F_n, say $\bigcup_i (E_n + a_j)$. Then $a_0 = 0$, while $a_{j+1} - a_j = n^2 c_n + n^{-1/2} c_n$. In different terms, the final term in each translate becomes $x(0)$ in its successor to the right. The number of translates is to be $[c_{n-1} c_n^{-1} n^{-13/6}]$ for $n \geqq 1$.

(a) In F_n we have the inequalities

$$c_n \leqq x(m+1) - x(m) \leqq (n^2 + n^{3/2}) c_n < 2n^2 c_n.$$

Thus E_n has diameter $< 2n^{-1/6} c_{n-1}$. The vector sum $E = \sum_{n=1}^\infty E_n$ is then an L-set. (It is somewhat easier to verify that, for large enough r, the subset

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$E=\sum_{n=r}^{\infty} E_n$ is an L-set; this would serve just as well.) In each E_n we construct the uniform probability distribution μ_n and then the convolution product $\prod_1 \mu_n$, a probability in E. To prove the theorem, we demonstrate in fact that

$$\lim_{u\to+\infty} \int \exp(iu\varphi(t))\mu(dt) = 0 \quad \text{for each } \varphi \text{ in } C^1_\omega.$$

To each large u we choose an n and observe

$$\left| \int \exp(iu\varphi(t))\mu(dt) \right| \leqq \sup_{s\in E} \left| \int \exp iu\varphi(t+s)\mu_n(dt) \right|.$$

Certain exponents u are handled without using the special properties of φ; the remaining exponents require more careful estimation. (In our second theorem we exploit this idea by making all exponents of the first sort.)

(b) $n^{1/5}c_{n-1}^{-1} \leqq u \leqq n^{-1/5}c_n^{-1}$. We must estimate the sums $\sum_{E_n} \exp(iu\varphi(x+s))$, $s\in E$, uniformly. Now for successive elements z_i and z_{i+1} of E_n we have

$$|z_{i+1} - z_i - c_n| \leqq 3n^{-3/2}c_n = o(c_n),$$
$$|u\varphi(z_{i+1} + s) - u\varphi(z_i + s) - u\varphi'(s)c_n| = o(uc_n).$$

The last relation requires only that φ' be bounded and uniformly continuous on an interval about E. Now $uc_n \leqq n^{-1/5}$, and the linear length of the sequence $\{u\varphi(x+s), x\in E\}$ is asymptotically $u\varphi'(s)c_n \cdot c_{n-1}c_n^{-1}n^{-1/6} \geqq \varphi'(s)n^{1/30} \to +\infty$. Thus this part of the argument can be concluded by geometrical reasoning concerning uniform distribution *modulo* 2π.

(c) For the remaining exponents we define n by the inequaltiy $n^{-1/5}c_n^{-1} < u < (n+1)^{1/5}c_n^{-1}$. Again we have recourse to uniform distribution, but first we split E_n into its constituents a_i+F_n, and then split F_n into residue classes *modulo n*. Thus we are attempting to estimate the distribution of sequences

$$u\varphi(s + x(m)), \qquad 1 \leqq m \leqq n,\ m \equiv r \ (\mathrm{mod}\ n).$$

Writing $y(p)=x(r+np), 0 \leqq p < n$, $1 \leqq r \leqq n$, we have sequences $u\varphi(y(p)+s)$, $0 \leqq p < n$. To these sequences we apply an inequality of van der Corput [**1**, pp. 71–73] and conclude that it will be sufficient to obtain the uniform distribution of the difference sequences

$$u\varphi(y(p + h) + s) - u\varphi(y(p) + s), \qquad 0 \leqq p < n - h,$$

for $h=1, 2, 3, \cdots$. (This need not be uniform with respect to h.) Now F_n has diameter $<4n^2c_n$ so

$$\varphi(y(p + h)) - \varphi(y(p) + s) = [y(p + h) - y(p)]\varphi'(y(0) + s) + \theta(y(p + h) - y(p))\omega(n^2c_n), \qquad |\theta| \leqq 4.$$

The error term can be majorized by

$$4\cdot 4nhc_n\cdot\omega(c_{n-1}) = O(n^{-1}hc_n) = o(u^{-1}).$$

Therefore the error term can be neglected, as can the factor $\varphi'(y(0)+)s$ in the remaining argument. Then

$$\begin{aligned} y(p+h)-y(p) &= hnc_n + [h^2n^2 + 2hn(r+pn)]n^{-5/2}c_n \\ &= A(h,n,r) + 2hn^{-1/2}c_n p. \end{aligned}$$

Here $A(h, n, r)$ depends only on the variables indicated. Thus

$$uy(p+h) - uy(p) = A' + 2uhn^{-1/2}c_n p,$$

with

$$un^{-1/2}c_n < (n+1)^{1/5}n^{-1/2} \to 0,$$

$$un^{-1/2}c_n\cdot n > n^{1/2}n^{-1/5} \to +\infty, \quad \text{and} \quad h \geqq 1.$$

The last two relations suffice for our purpose, since p assumes the values in $[0, n-1]$. Thus the exceptional exponents u are disposed of, and the proof is complete.

In our second theorem we consider all C^1 maps from a rectangle in R^2 to a Euclidean space R^m ($m\geqq 1$). All maps except a set of the first category transform a certain set of uniqueness onto an M_0-set.

The theorem does not require Baire's theorem to demonstrate the existence of the C^1 map, since maps with polynomial coefficients can be written explicitly.

Let S_1 and S_2 be sets of positive integers, each containing segments of unbounded length, and highly disjoint in the following sense: to each K the inequality $|s_1-s_2|<K$ ($s_i\in S_i$) has only a finite number of solutions s_1, s_2. Then E_i is the set of sums $\sum_{n\notin S_i}\varepsilon_n 2^{-n}$ ($\varepsilon_n=0, 1$), and so E_i is an L-set. In E_i we place the canonical product measure and on $E=E_1\times E_2$ the measure $\mu=\mu_1\times\mu_2$.

Definition. A measurable function h on $E_1\times E_2$ to R^m is called *projectively diffuse* provided $\mu\{z:h(z)\in V\}=0$ for every vector subspace $V\neq R^m$. Equivalently, h is projectively diffuse provided

$$\lim_{\varepsilon\to 0}\mu\{z:|(h(z), u)| < \varepsilon\,\|u\|\}$$

uniformly for all u in R^m.

Theorem II. (i) *Let $F(x,y)$ be a C^1 map of R^2 into R^m such that $\partial F/\partial x$ and $\partial F/\partial y$ are projectively diffuse. Then $F(E)$ is an M_0-set in R^m:*

$$\lim\int \exp i(u,F)\,d\mu = 0 \quad \textit{as } \|u\|\to\infty \textit{ in } R^m.$$

(ii) *Moreover, these mappings form a set of second category in the B-space $C^1(I; R^m)$, where I is a closed rectangle containing E, and μ is an arbitrary diffuse measure on E.*

It is easy to write down functions F, relative to measures $\mu=\mu_1\times\mu_2$, provided only that each factor is a diffuse measure. Let $e_1, \cdots, e_m$ be a basis for R^m and let $F(x, y)=\sum e_\kappa(x+y)^\kappa$. Then for any linear form $l\neq 0$, $l(\partial F/\partial x)=l(\partial F/\partial y)$ has only a finite number of zeroes on any line, so its zero-set is $\mu_1\times\mu_2$-null.

PROOF OF THEOREM II (i). Let E_n denote any of the subsets of E determined by a choice of the first n coordinates in the factors E_1 and E_2, and let Q_n denote the closed convex hull of E_n. Then to each $\delta>0$ there is an $\varepsilon>0$ and integer N with this property: for every element u of norm 1 in R^m, the squares Q_N meeting the set $\{|(\partial_x F, u)|<\varepsilon$ or $|(\partial_y F, u)|<\varepsilon\}$ have total μ-measure at most δ. We call the remaining rectangles Q_N *admissible* for u; they are disjoint except for a set of μ-measure 0. Now let $u=\|u\|u_0$; to prove Theorem II (i) it suffices to prove that

$$\lim\int_{Q_N} \exp i(u, F)\, d\mu = 0, \quad Q_N \text{ admissible for } u_0.$$

Indeed, for each u_0 the admissible rectangles form a disjoint family of total measure $>1-\delta$. The restriction of μ to Q_N is easily described; let μ_1^N and μ_2^N be the product measures on $\{\sum \varepsilon_n 2^{-n};\, n\notin S_i,\, n>N\}$ and $\lambda^N=\mu_1^N\times\mu_2^N$. Then the restriction is obtained from λ^N by a translation and a scalar multiplication. Therefore our problem is reduced to estimating integrals

$$\int \exp i\, \|u\|\, (u_0, F(z+z^*))\lambda^N(dz), \qquad z^*\in Q_N.$$

Suppose for definiteness that $\log\|u\|/\log 2$ is further from S_1 than from S_2. The integral is reduced to an iterated integral

$$\iint \exp i\, \|u\|\, G(x, y)\mu_1^N(dx)\mu_2^N(dy),$$

where $|\partial_x G|>\varepsilon$ on a rectangle containing the support of the measure, and the metric properties of $\partial_x G$ are no worse than those of $\partial_x F$. Now μ_1^N contains as a factor the uniform distribution on a set $\{\sum \varepsilon_n 2^{-n}: r\leqq n\leqq p\}$, namely an arithmetic progression of difference 2^{-p}, and 2^{p-r+1} terms. We can attain $\log\|u\|/\log 2-r\to+\infty$, $p-\log\|u\|/\log 2\to+\infty$, that is $2^{-p}\|u\|\to 0$, $2^{-r}\|u\|\to 0$. Since the progression has length on (the real line) 2^{-r+1}, the proof can now be completed as in Theorem I.

PROOF OF THEOREM II (ii). For any diffuse measure μ, let N_μ be the set of F in $C^1(I; R^m)$ for which there is a linear form $l\neq 0$ so that

$\mu\{z:\partial_x l(F(z))=0\}>0$; thus N_μ is an F_σ in C^1. Further, if $\mu=\sum_{j=1}^{\infty}\lambda_j$ is expressed as a sum of positive measures, then $N_\mu=\bigcup_j N_{\lambda_j}$. Polynomials $P(x,y)$ are dense in C^1 and by the device used after the statement of the theorem, we see that the set $\{p:\partial_x l(p)\neq 0$ for all forms $\neq 0\}$ is dense in C^1. Then N_μ has void interior in C^1 unless $\mu(Z_1)>0$, Z_1 being the zero-set of some polynomial $p_1\neq 0$. Writing $\mu(X)\equiv\mu(X\cap Z_1)+\mu(X\sim Z_1)\equiv\lambda_1(X)+\lambda_2(X)$, we can iterate this for the measure $\lambda_2,\cdots$. Thus $\mu=\lambda+\sum_{j=1}^{\infty}\lambda_j$ where N_λ has void interior and λ_j is concentrated on a zero-set Z_j. Next we observe that by the implicit function theorem each Z_j is a finite or countable union of analytic images of $(0, 1)$; we can therefore conclude that a polynomial having an uncountable number of zeroes on Z_j vanishes identically on each connected component of Z_j. To any m distinct points on an infinite component, Z_k, there is a dense set of polynomials p so that $\{\partial_x p(Z_k)\}_1^m$ has rank m and hence for any $l\neq 0$, $l(\partial_x p)$ has only isolated zeroes on the component. Because Z_j is a countable union of its components, λ_j is a sum of measures for which the exceptional sets are nondense, and the theorem is proved.

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