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DIRICHLET FINITE SOLUTIONS OF $\Delta u = Pu$

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ABSTRACT. The purpose of this paper is to give a necessary and also a sufficient condition for a Dirichlet finite harmonic function on a Riemann surface to be represented as a difference of a Dirichlet finite solution of $\Delta u = Pu$ ($P \geqq 0$) and a Dirichlet finite potential of signed measure.

1. Let $P = P(z)\,dx\,dy$ ($z = x + iy$) be a nonnegative not identically zero α-Hölder continuous ($0 < \alpha \leqq 1$) second order differential on a Riemann surface R and $\mathrm{PD}(R)$ be the Hilbert space of all Dirichlet finite solutions of

$$(1) \qquad \Delta u(z) = P(z)u(z), \qquad \Delta \cdot = 4\partial^2 \cdot / \partial z \partial \bar{z},$$

on R with the scalar product given by mixed Dirichlet integral, i.e. $(u, v) = D_R(u, v) = \int_R du \wedge {}^*dv$, *not* the energy integral.

The study of $\mathrm{PD}(R)$ was begun by Royden [**6**]. We will use the fact shown by Nakai [**2**] that $\mathrm{PD}(R)$ forms a vector lattice under the natural order in $\mathrm{PD}(R)$. We also use the Glasner-Katz maximum principle [**1**] that the modulus of every function in $\mathrm{PD}(R)$ takes its maximum on the Royden harmonic boundary. The recent result of Nakai [**3**] that $\mathrm{PBD}(R)$ is dense in $\mathrm{PD}(R)$ will not be made use of.

Let $\Delta(R)$ be the Royden harmonic boundary and $\mathrm{HD}(R)$ be the class of Dirichlet finite harmonic functions on R. (For the basic materials from the Royden compactification and the class $\mathrm{HD}(R)$ we refer to the monograph of Sario and Nakai [7].) One of the important problems in the theory of $\mathrm{PD}(R)$ which is not fully developed yet is to describe the distribution of $\mathrm{PD}(R)|\Delta(R)$ in $\mathrm{HD}(R)|\Delta(R)$. We will prove a theorem which contributes to this question.

2. If R is parabolic, then $\mathrm{PD}(R) = \{0\}$ (cf. Royden [**6**]), which case offers no interest. Therefore we assume throughout the paper that R is hyperbolic. Let $\tilde{M}(R)$ be the class of all Dirichlet finite Tonelli functions on R and $\tilde{M}_\Delta(R)$ the subclass of $\tilde{M}(R)$ consisting of functions f with $f|\Delta(R) = 0$ (cf. [7]). We then have the orthogonal decomposition

$$\tilde{M}(R) = \mathrm{HD}(R) + \tilde{M}_\Delta(R),$$

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and since $\mathrm{PD}(R) \subset \tilde{M}(R)$, we can define an operator $T: \mathrm{PD}(R) \to \mathrm{HD}(R)$ characterized by

$$u - Tu \in \tilde{M}_{\Delta}(R). \tag{2}$$

Using results cited in §1 we can show that T is a vector space isomorphism from $\mathrm{PD}(R)$ onto

$$X_D(R) \equiv T(\mathrm{PD}(R)) \tag{3}$$

such that $u > 0$ is equivalent to $Tu > 0$ and $\sup_R |u| = \sup_R |Tu|$. Therefore the study of $\mathrm{PD}(R)$ can be reduced to that of $X_D(R)$ and for this reason we call T the *reduction operator* for Dirichlet finite solutions. It can be seen that

$$u = Tu - \frac{1}{2\pi} \int_R G(\cdot, \zeta) P(\zeta) u(\zeta) \, d\xi \, d\eta \qquad (\zeta = \xi + i\eta)$$

(cf. [**3**]). We will discuss when $h \in \mathrm{HD}(R)$ belongs to $X_D(R)$.

3. Let Ω be a regular subregion of R. By the *P-unit* e_Ω on Ω we mean the solution e_Ω of (1) on Ω with the continuous boundary values 1. The net $\{e_\Omega\}$ for every regular subregion Ω is decreasing and hence convergent to a solution on R:

$$e_R = \lim_{\Omega \to R} e_\Omega \geqq 0$$

which we call the P-unit on R. The only bounded solution of (1) on R is zero if and only if $e_R \equiv 0$ (Ozawa [**5**], Royden [**6**]). We describe $X_D(R)$ in terms of $\{e_\Omega\}$ and e_R as follows:

THEOREM. *Suppose that $h \in \mathrm{HD}(R)$. If $h \in X_D(R)$, then*

$$D_R(e_R h) < \infty. \tag{4}$$

Conversely if

$$\limsup_{\Omega \to R} D_\Omega(e_\Omega h) < \infty, \tag{5}$$

then $h \in X_D(R)$.

The proof will be given in §§4 and 7. The condition (4) is necessary for $h \in X_D(R)$ but not sufficient. In fact, let $R = \{|z| < 1\}$ and $P(z) = 4(1 + |z|^2)(1 - |z|^2)^{-2}$. Then $e_R \equiv 0$ and $X_D(R) = \{0\}$ (Royden [**6**]), while (4) is trivially valid for every $h \in \mathrm{HD}(R)$. The condition (5) is sufficient for $h \in X_D(R)$ but not necessary. We exhibit an instructive example due to Nakai [**4**]. Let $R = \{|z| > 1\}$ and $P(z) = 1 + |z|^{-1}$. Consider $\Omega_n = \{1 + n^{-1} < |z| < n\}$ $(n = 2, 3, \cdots)$ which exhausts R as $n \to \infty$. Denote by e_n the P-unit on Ω_n. The P-unit e_R on R is given by

$$e_R(z) = \left(e \int_1^\infty e^{-2t} t^{-1} \, dt \right)^{-1} \cdot e^{|z|} \cdot \int_{|z|}^\infty e^{-2t} \cdot t^{-1} \, dt$$

and a straightforward calculation shows that $e_R \in \mathrm{PD}(R)$ and hence $1 \in X_D(R)$. We also see that

$$e_n(z) = \alpha_n e^{|z|} + \beta_n e^{|z|} \int_1^{|z|} e^{-2t} t^{-1}\, dt$$

where

$$\alpha_n = e^{-n} - (e^{-n} - e^{-(1+1/n)}) \left(\int_{1+1/n}^{n} e^{-2t} t^{-1}\, dt \right)^{-1} \int_1^n e^{-2t} t^{-1}\, dt,$$

and

$$\beta_n = (e^{-n} - e^{-(1+1/n)}) \left(\int_{1+1/n}^{n} e^{-2t} t^{-1}\, dt \right)^{-1}.$$

However an easy but cumbersome computation shows that

$$D_R(1 \cdot e_n) = \mathcal{O}(n) \qquad (n \to \infty)$$

and (5) is not valid for $h = 1 \in X_D(R)$. By Fatou's lemma, (5) implies (4). That the converse is not necessarily true is also seen from the above example.

4. **Necessity of (4).** Suppose $h \in X_D(R)$. Since $X_D(R)$ forms a vector lattice along with $\mathrm{PD}(R)$, we may assume $h > 0$ to prove (4). Let $u \in \mathrm{PD}(R)$ such that $h = Tu$ and let $\varphi = u - h$. We will prove (4) both for u and φ.

We write $\|\cdot\|_\Omega = (D_\Omega(\cdot))^{1/2}$. By Green's formula

$$\begin{aligned}
\|u(1 - e_\Omega)\|_\Omega^2 &= -\int_\Omega u(1 - e_\Omega) d\, {}^*d(u(1 - e_\Omega)) \\
&= -\int_\Omega u^2(1 - e_\Omega)^2 P + \int_\Omega u^2(1 - e_\Omega) e_\Omega P \\
&\quad + 2\int_\Omega u(1 - e_\Omega)\, du \wedge {}^*de_\Omega \\
&\leqq \int_\Omega u(1 - e_\Omega) Pu + 2\int_\Omega (1 - e_\Omega)\, du \wedge {}^*(u\, de_\Omega) \\
&= \int_\Omega u(1 - e_\Omega) d\, {}^*du + 2\int_\Omega (1 - e_\Omega)\, du \wedge {}^*(d(ue_\Omega) - e_\Omega\, du).
\end{aligned}$$

Observe that $\int_\Omega u(1 - e_\Omega) d\, {}^*du = -\int_\Omega d(u(1 - e_\Omega)) \wedge {}^*du$. By Schwarz's inequality

$$\|u(1 - e_\Omega)\|_\Omega^2 \leqq \|u(1 - e_\Omega)\|_\Omega \|u\|_\Omega + 2\, \|u\|_\Omega \|ue_\Omega\|_\Omega + 2\, \|u\|_\Omega^2.$$

In view of $\|ue_\Omega\|_\Omega \leqq \|u(1 - e_\Omega)\|_\Omega + \|u\|_\Omega$, we deduce

$$\|u(1 - e_\Omega)\|_\Omega^2 \leqq 3\, \|u(1 - e_\Omega)\|_\Omega \cdot \|u\|_\Omega + 4\, \|u\|_\Omega^2.$$

This implies $\|u(1 - e_\Omega)\|_\Omega \leqq 4\|u\|_\Omega$ or $\|ue_\Omega\|_\Omega \leqq 5\|u\|_\Omega$. Therefore, by Fatou's lemma,

$$D_R(e_R u) \leqq \liminf_{\Omega \to R} D_\Omega(e_\Omega u) \leqq 25 D_R(u) < \infty.$$

5. Let $h_\Omega \in C(\bar{\Omega})$ such that h_Ω is harmonic in Ω and $h_\Omega | \partial\Omega = u$. Set $\varphi_\Omega = u - h_\Omega$. Observe $\Delta\varphi_\Omega = Pu$ and $\varphi_\Omega \leqq 0$. Since $D_\Omega(u) = D_\Omega(h_\Omega) + D_\Omega(\varphi_\Omega)$ and $\lim_{\Omega\to R} h_\Omega = h$, we infer that $\varphi = \lim_{\Omega\to R} \varphi_\Omega$, $d\varphi = \lim_{\Omega\to R} d\varphi_\Omega$, and $D_\Omega(\varphi_\Omega) \leqq D_\Omega(u)$. By Green's formula,

$$\begin{aligned} \|e_\Omega\varphi_\Omega\|_\Omega^2 &= -\int_\Omega e_\Omega\varphi_\Omega d\,{}^*d(e_\Omega\varphi_\Omega) \\ &= -\int_\Omega e_\Omega^2\varphi_\Omega^2 P - \int_\Omega e_\Omega^2 \varphi_\Omega u P - 2\int_\Omega e_\Omega\varphi_\Omega\, de_\Omega \wedge {}^*d\varphi_\Omega \\ &\leqq -\int_\Omega \varphi_\Omega d\,{}^*du - 2\int_\Omega e_\Omega\, d\varphi_\Omega \wedge {}^*(\varphi_\Omega\, de_\Omega) \\ &= \int_\Omega d\varphi_\Omega \wedge {}^*du - 2\int_\Omega e_\Omega\, d\varphi_\Omega \wedge {}^*(d(e_\Omega\varphi_\Omega) - e_\Omega\, d\varphi_\Omega). \end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned} \|e_\Omega\varphi_\Omega\|_\Omega^2 &\leqq \|\varphi_\Omega\|_\Omega \|u\|_\Omega + 2\,\|\varphi_\Omega\|_\Omega \|e_\Omega\varphi_\Omega\|_\Omega + 2\,\|\varphi_\Omega\|_\Omega^2 \\ &\leqq 2\,\|u\|_\Omega \|e_\Omega\varphi_\Omega\|_\Omega + 3\,\|u\|_\Omega^2 \end{aligned}$$

and therefore $\|e_\Omega\varphi_\Omega\|_\Omega \leqq 3\,\|u\|_\Omega$. By Fatou's lemma we deduce

$$D_R(e_R\varphi) \leqq \liminf_{\Omega\to R} D_\Omega(e_\Omega\varphi_\Omega) \leqq 9D_R(u) < \infty.$$

6. **Sufficiency of (5).** Let $u_\Omega \in C(\bar{\Omega})$ such that $\Delta u_\Omega(z) = P(z)u_\Omega(z)$ on Ω and $u_\Omega | \partial\Omega = h$. By Green's formula,

$$\begin{aligned} \|u_\Omega - he_\Omega\|_\Omega^2 &= -\int_\Omega (u_\Omega - he_\Omega) d\,{}^*d(u_\Omega - he_\Omega) \\ &= -\int_\Omega (u_\Omega - he_\Omega)u_\Omega P + \int_\Omega (u_\Omega - he_\Omega)he_\Omega P \\ &\quad + 2\int_\Omega (u_\Omega - he_\Omega)\, dh \wedge {}^*de_\Omega \\ &= -\int_\Omega (u_\Omega - he_\Omega)^2 P + 2\int_\Omega (u_\Omega - he_\Omega)\, dh \wedge {}^*de_\Omega \\ &\leqq 2\int_\Omega dh \wedge {}^*(d(e_\Omega u_\Omega) - e_\Omega\, du_\Omega) \\ &\quad -2\int_\Omega e_\Omega\, dh \wedge {}^*(d(he_\Omega) - e_\Omega\, dh). \end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned} \|u_\Omega - he_\Omega\|_\Omega^2 &\leqq 2\,\|h\|_\Omega \|e_\Omega u_\Omega\|_\Omega + 2\,\|h\|_\Omega \|u_\Omega\|_\Omega \\ &\quad + 2\,\|h\|_\Omega \|he_\Omega\|_\Omega + 2\,\|h\|_\Omega^2. \end{aligned}$$

By the same estimate as in §4, we deduce $\|e_\Omega u_\Omega\|_\Omega \leqq 5\|u_\Omega\|_\Omega$ and $\|u_\Omega\|_\Omega \leqq \|u_\Omega - he_\Omega\|_\Omega + \|he_\Omega\|_\Omega$, and hence

$$\|u_\Omega - he_\Omega\|_\Omega^2 \leqq 12\ \|h\|_\Omega\ \|u - he_\Omega\|_\Omega + 14\ \|h\|_\Omega\ \|he_\Omega\|_\Omega + 2\ \|h\|_\Omega^2.$$

By (5) we conclude that

$$D_\Omega(u_\Omega) \leqq K < \infty \tag{6}$$

for every Ω with a constant K.

7. Fix an Ω_0 such that $P \not\equiv 0$ on Ω_0. Since $\mathrm{PD}(\Omega_0)$ is a Hilbert space with reproducing kernel (cf. [**2**]), (6) implies that there exists an exhaustion $\{\Omega_n\}$ of R with $\Omega_n \supset \Omega_0$ such that $\{u_{\Omega_n}\}$ converges uniformly on each compact set of Ω_0. By a diagonal process, we may assume that $\{u_{\Omega_n}\}$ converges uniformly on each compact set of R. Let $u = \lim_{n\to\infty} u_{\Omega_n}$. Because of (6) and Fatou's lemma we see that $u \in \mathrm{PD}(R)$. We can regard $h - u_{\Omega_n}$ as an element of $\tilde{\boldsymbol{M}}_0(R) \subset \tilde{\boldsymbol{M}}_\Delta(R)$. Since $\lim_{n\to\infty} (h - u_{\Omega_n}) = h - u$ uniformly on each compact set of R and $\sup_n D_R(h - u_{\Omega_n}) < \infty$, Kawamura's lemma (cf. [**7**]) implies that $h - u \in \tilde{\boldsymbol{M}}_\Delta(R)$, i.e. $h = Tu$ and a fortiori $h \in X_D(R)$.

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