BI-UNITARY PERFECT NUMBERS

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Abstract. Let d be a divisor of a positive integer n. Then d is a unitary divisor if d and n/d are relatively prime, and d is a bi-unitary divisor if the greatest common unitary divisor of d and n/d is 1. An integer is bi-unitary perfect if it equals the sum of its proper bi-unitary divisors. The purpose of this paper is to show that there are only three bi-unitary perfect numbers, namely 6, 60 and 90.

A divisor d of an integer n is a unitary divisor if d and n/d are relatively prime. A divisor d of an integer n is a bi-unitary divisor if the greatest common unitary divisor of d and n/d is 1. Let \( \sigma(n) \) be the sum of the divisors of n, let \( \sigma^*(n) \) be the sum of the unitary divisors of n, and let \( \sigma^{**}(n) \) be the sum of the bi-unitary divisors of n.

We say that N is unitary perfect if \( \sigma^*(N)=2N \). Subbarao and Warren [2] showed that 6, 60, 90 and 87360 are the first four unitary perfect numbers; Wall reported [3] that

\[
146,361,946,186,458,562,560,000 = 2^{183} \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313
\]

is also unitary perfect and later showed [4] it to be the next such number after 87360. Subbarao [1] has conjectured that there are only finitely many unitary perfect numbers.

We say that N is bi-unitary perfect if \( \sigma^{**}(N)=2N \). The purpose of this paper is to show that the first three unitary perfect numbers, i.e., 6, 60 and 90, are the only bi-unitary perfect numbers.

One easily verifies that \( \sigma^{**} \) is multiplicative and that if p is prime and \( e \geq 1 \), then

\[
\sigma^{**}(p^e) = \sigma(p^e) = (p^{e+1} - 1)/(p - 1)
\]

if \( e \) is odd, and

\[
\sigma^{**}(p^e) = (p^{e+1} - 1)/(p - 1) - p^{e/2}
\]

if \( e \) is even. Hence \( \sigma^{**}(n) \leq \sigma(n) \) with equality if and only if every prime which divides n does so an odd number of times. It also follows immediately that \( \sigma^{**}(n) \) is odd if and only if n is 1 or a power of 2; consequently, each odd prime power unitary divisor of n contributes at least one factor 2 to \( \sigma^{**}(n) \).

Received by the editors June 10, 1971.

AMS 1970 subject classifications. Primary 10A20; Secondary 10A99.

Key words and phrases. Bi-unitary divisors, perfect numbers.
Let \( \phi \) be Euler's totient.

**Theorem 1.** There are no odd bi-unitary perfect numbers.

**Proof.** If \( \sigma^{**}(N) = 2N \) and \( N \) is odd, then \( N \) must be a prime power, say \( N = p^e \). But then

\[
\sigma^{**}(p^e)/p^e \leq \sigma(p^e)/p^e < p^e/\phi(p^e) = p/(p-1) \leq 2/3,
\]

so \( N \) cannot be bi-unitary perfect.

We define \( f(n) = \sigma^{**}(n)/n \) and note that if \( p \) is any prime, then

\[
1 = f(1) < f(p^2) < f(p) < f(p^3) < f(p^9),
\]

and

\[
f(p^9) < f(p^{*+3}) < p/(p - 1)
\]

for all natural numbers \( \alpha \), and \( f(p^{*+3}) > f(p^{*+1}) \) if \( \alpha \) is odd. Also, we remark that \( f(N) = 2 \) if and only if \( N \) is bi-unitary perfect.

**Theorem 2.** The only even bi-unitary perfect numbers are 6, 60 and 90.

**Proof.** Henceforth we assume that \( N \) is bi-unitary perfect and even, and write \( N = 2^a M = 2^a XY \) where \( X = \sigma^{**}(2^a) \). Our scheme of proof is to establish that: (i) if \( a \) is odd, then \( a = 1 \); (ii) if \( a = 1 \), then \( N = 6 \) or \( N = 90 \); (iii) if \( a = 2 \), then \( N = 60 \); (iv) if \( a \geq 6 \), then \( (N, 3) = 1 \); (v) \( a = 4 \) and \( a = 6 \) are impossible; and (vi) \( a \geq 8 \) is impossible.

If \( a \) is odd, then \( 3 \nmid X \). If \( a \geq 3 \), then \( f(2^a) \geq \frac{1}{a} \), so

\[
f(N) \geq f(8)f(9) = (\frac{1}{8})(\frac{1}{9}) = \frac{8}{9} > 2,
\]

contradicting the assumption that \( f(N) = 2 \). Thus (i) is proved.

If \( a = 1 \), then \( 3 \mid N \) and \( N \) has at most two distinct odd prime divisors. If \( 3 \nmid N \), then

\[
f(N) \geq (\frac{3}{2})(\frac{1}{4}) = \frac{3}{8} > 2.
\]

If \( 3 \nmid N \), then \( N = 6 \). If \( 3 \nmid N \), then \( 5 \mid N \) as \( 1 + 3^2 = 2 \cdot 5 \) and 5 cannot divide \( N \) exactly twice without \( N \) having three distinct odd prime divisors. Then \( f(N) > 2 \) unless \( 5 \mid N \), which yields \( N = 90 \). Thus (ii) is proved.

If \( a = 2 \), then \( 5 \mid N \) and \( N \) has at most three distinct odd prime divisors. Suppose \( 3 \nmid N \), and let \( N = 4M \) with \( M \) odd. Then since

\[
f(N) < 5M/4\phi(M)
\]

and

\[
(5 \cdot 7 \cdot 11 \cdot 13)/(4 \cdot 6 \cdot 10 \cdot 12) < (5 \cdot 5 \cdot 11 \cdot 13)/(4 \cdot 4 \cdot 10 \cdot 12)
\]

\[
< (5 \cdot 5 \cdot 7 \cdot 13)/(4 \cdot 4 \cdot 6 \cdot 12) < 2,
\]

we must have \( N = 2^5 b^7 c^d \) for some choice of positive exponents \( b \), \( c \) and \( d \). Moreover, \( A = \sigma^{**}(11^d) \) must be divisible by 2 exactly once and
not by 3, which requires that \(d\) be even, say \(d=2e\). Then

\[
A = \frac{(1 + 11^e-1)(11^e - 1)}{10}
\]

and 5 and 7 are the only odd primes which can divide \(A\). From congruences modulo 7, it is clear that if \(7\nmid A\) then \(3\mid e\); but then the factor \((11^e-1)\) is a multiple of 19, so \(19\nmid A\), a contradiction. Now, \(A\) cannot be a power of 2, because to avoid having \(1 + 11^e+1\) be a multiple of 3, we must have \(e\) odd, whence \((11^e - 1)/10\) is odd. If \(7\nmid A\), then \(5\nmid A\), so \(5^3(11^e - 1)\) and consequently \(5\mid e\); but then \(3221\nmid A\), a contradiction. Thus \(3\nmid A\).

If \(a=2\) and \(3\nmid N\) we write \(N=2^33^55^7N'\) where \((N', 30)=1\); in fact, \(N'\) is either 1 or a prime power. If neither \(b\) nor \(c\) is 2 then

\[
f(N) \geq \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) > 2;
\]

if \(b\) is 1 or 3 there are too many factors 2 in the numerator of \(f(N)\), and if \(b=4\), there is an excess factor 7; if \(b=2\), then

\[
f(N) \leq \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) < 2.
\]

Thus we cannot have \(c=2\). If \(b=2\), then as \(5^3\mid \sigma^{**}(2^33^5)\) we must have \(c\geq 2\). So \(c\geq 3\) as the case \(b=c=2\) has already been eliminated. If \(7\nmid N\) then

\[
f(N) < \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) < 2.
\]

Thus, \(7\nmid N\). If \(c=4\), then \(N\) has too many factors 3. The possibilities \(7\nmid N\), \(7^3\nmid N\) and \(7^4\nmid N\) lead to too many factors 2 in \(N\), while \(7^2\nmid N\) implies that \(f(N)<2\). Therefore, \(7^5\nmid N\) and if \(c=3\) or \(c\geq 5\), then

\[
f(N) \geq \left(\frac{1}{2}\right)\left(\frac{1}{7}\right)f(5^77^5) > 2.
\]

Hence (iii) is proved.

If \(N\) is divisible by \(3\cdot 2^a\) then

\[
f(N) \geq \left(\frac{1}{3}\right)\left(\frac{1}{2^a}\right)\left(\frac{1}{7}\right) > 2,
\]

contradicting the fact that \(f(N)=2\). Thus (iv) is proved.

If \(a=4\), then \(N=16\cdot 27N'\) with \(N'\) odd. Then

\[
f(N) \geq \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{7}\right) > 2,
\]

a contradiction. If \(a=6\), then \(7\cdot 17\nmid M\). If 7 does not divide \(M\) exactly twice, then \(f(N)>2\); thus \(7^a\nmid M\), so \(5^b\nmid M\). If \(5^b\nmid M\) then \(f(N)>2\). Hence
5^a || M, so 13|M and 13 does not divide M exactly twice or else 5^3|M. Then
\[ f(N) \geq (\frac{1}{2^a})(\frac{3}{4})(\frac{5}{6})(\frac{7}{8}) > 2, \]
a contradiction which establishes (v).

If \( a \geq 8 \), then \((N, 5) = 1\) or else \( f(N) \geq (\frac{5}{6})(\frac{7}{8}) > 2\), a contradiction.
We set \( a = 2b \) with \( b \geq 4 \). Then
\[ X = \sigma^*(2^b) = 2^{2b+1} - 2^b - 1 = (2^b - 1)(1 + 2^{b+1}) \]
is composite. Since \((3, M) = 1\) and \( X|M \), and \( 1 + 2^{b+1} = 2(2^b - 1) + 3 \), we know the factors \( 2^b - 1 \) and \( 1 + 2^{b+1} \) are relatively prime.

Let \( p \) be any prime dividing \( X \), and suppose \( p^e || M \). If \( c \neq 2 \), then
\[ 1 + p^{-1} \leq f(p^e) \leq f(M) = 2^{2b+1}/(2^{2b+1} - 2^b - 1), \]
which requires that \( p \geq 2^{b+1} - 3 + 2/(1 + 2^b) > X^{1/2} \), whence \( p = X \), contradicting the fact that \( X \) is composite. Thus any prime that divides \( X \) must divide \( M \) exactly twice. Then
\[ 1 + p^{-2} = f(p^e) \leq f(M) \]
requires that \( p^e > X \). Hence \( X \) has no more than three distinct prime factors, and \((X, 30) = 1\). But \( X = (2^b - 1)(1 + 2^{b+1}) \), so one of the factors must be prime. As \((X, 3) = 1\), we must have \( b \) odd. With the restriction that \( b \) be odd, either
\[ 2^b - 1 \equiv 1 \pmod{10} \quad \text{and} \quad 1 + 2^{b+1} \equiv 5 \pmod{10} \]
or
\[ 2^b - 1 \equiv 1 + 2^{b+1} \equiv 7 \pmod{10}. \]
The first case is eliminated as \((N, 5) = 1\). In the second case, one of the two numbers must be prime: call this prime \( p \). Then \( p \equiv 7 \pmod{10} \) and \( p^a || N \). But \( \sigma^*(p^a) = 1 + p^a \) is then a multiple of 5, so \( 5 || N \), a contradiction.

Hence the theorem is proved.

The author thanks the referee for providing a portion of part (iii) of the proof of Theorem 2.

REFERENCES


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