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# CYCLIC VECTORS OF A CYCLIC OPERATOR SPAN THE SPACE

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ABSTRACT. If a bounded linear operator on a Banach space has a cyclic vector, then the set of cyclic vectors spans the space.

In their paper [1], B. Sz.-Nagy and C. Foiaş proved, as a corollary of a result on unitary dilations, that if a (linear bounded) operator $T$ on a *Hilbert space* has a cyclic vector then the set of all cyclic vectors of $T$ spans the whole space.

Here we give an entirely elementary proof of this theorem which also applies to operators on Banach spaces.[1]

THEOREM. *Let $T$ be an operator on the Banach space $X$. Suppose $x_0$ is a cyclic vector for $T$. Then, for any constant $c\neq 0$ such that $\|cT\|<1$, the vectors*

$$x_p = (I - cT)^p x_0 \qquad (p = 1, 2, \cdots)$$

*are also cyclic for $T$. The vectors $x_p$ $(p=0, 1, \cdots)$ span $X$ linearly.*

PROOF.[1] (1) Since $\|cT\|<1$, we have

$$I = (I - cT)^{-1}(I - cT) = \sum_{n=0}^{\infty} (cT)^n (I - cT) \quad \text{(in operator norm)},$$

hence

$$x_p = \sum_{n=0}^{\infty} c^n T^n x_{p+1},$$

and consequently

$$T^m x_p = \sum_{n=0}^{\infty} c^n T^{n+m} x_{p+1} \qquad (m = 0, 1, \cdots).$$

This shows that if $x_p$ is cyclic for $T$, then $x_{p+1}$ is cyclic for $T$ too. Thus all the $x_p$'s $(p\geqq 0)$ are cyclic.

(2) From the relation

$$(cT)^n = [I - (I - cT)]^n = \sum_{p=0}^{n} (-1)^p \binom{n}{p} (I - cT)^p \qquad (n = 0, 1, \cdots),$$

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[1] This proof was made in 1968 as mentioned in a footnote of [1].

it follows

$$T^n x_0 = \frac{1}{c^n} \sum_{p=0}^{n} (-1)^p \binom{n}{p} x_p \qquad (n = 0, 1, \cdots).$$

Since the set $\{T^n x_0\}_0^\infty$ spans $X$ by virtue of the cyclicity of $x_0$, we infer that the set $\{x_p\}_0^\infty$ also spans $X$.

This finishes the proof.

## Reference

1. B. Sz.-Nagy and C. Foiaş, *Vecteurs cycliques et quasi-affinités*, Studia Math. **31** (1968), 35–42. MR **38** #5050.

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