UNITARY GROUPS AND COMMUTATORS

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Abstract. If $H$ is a possibly unbounded selfadjoint operator and $A$ is a closed operator in a Hilbert space, the relation $(U_t^{-1}AU_t)f = iU_t^{-1}(AH-HA)U_t f$ can be shown to hold under relatively reasonable hypotheses on $A$ and $f$, where $U_t = e^{itH}$. This relation can then be used to relate properties of the commutator $AH-HA$ to properties of $A$ and $H$.

In quantum mechanics, a state $f$ at time $t=0$ evolves at time $t_0$ into the state $U_{t_0}f$, where $U_t = e^{itH}$ and $H$ is the quantum mechanical Hamiltonian operator for the system. This means that for the observable $A$, the expectation of $A$ in the state $U_{t_0}f$ is given by $(AU_{t_0}, U_{t_0}f)$. Equivalently, we may regard the state as fixed and the observable $A$ as evolving with time. Thus at time $t$ the new observable $A_t$ is $U_t^{-1}AU_t$. To analyze this evolution further, an obvious step is to differentiate with respect to $t$, which yields the formal relation $A'_t = iU^{-1}_t(AH-HA)U_t$. If $i(AH-HA)$ is positive definite, for example, this means that expectations are increasing.

Thus one is naturally led to study the commutator $AH-HA$. We shall use the group $U_t$ as an essential tool in our study, and the hypotheses of our theorems will explicitly involve $U_t$. This seems justified physically, since $U_t$ has direct physical significance.

A quite different method of relating $A$, $H$ and $AH-HA$ is given in the interesting book by Putnam [3].

In what follows, we let $U_t = e^{itH}$, and $H$ be a selfadjoint operator in a Hilbert space $h$. $A$ will be a closed operator in $h$. Take domain $H^n$ to mean the intersection of the domains of all $H^n$, where $n$ ranges over the positive integers. Take $H^0$ to be the identity operator.

We first state and prove conditions under which the relationship $(U_t^{-1}AU_t)f = iU_t^{-1}(AH-HA)U_t f$ holds.

Theorem 1. Let $n$ be a nonnegative integer, and let $m > n$ be a positive integer or $\infty$. Suppose that domain $A$ contains domain $H^n$, and that $A$ takes
domain $H^n$ into domain $H$. Then, for any $f$ in domain $H^n$, $(U^{-1}_t A U_j f)'$
exists in the strong sense and is equal to $i[U^{-1}_t (AH - HA) U_j] f$.

**Remark.** If $AH$ and $HA$ were both defined on domain $H^i$, for some
nonnegative integer $i$, the hypotheses of Theorem 1 would hold, taking
$n = i$, and $m = i + 1$.

**Proof.** We prove the theorem by taking difference quotients, after
first observing that $U_t$ takes domain $H^i$ onto itself, for any $i$ which is
either a nonnegative integer or $\infty$.

Now

$$U^{\Delta t}_t A U^{\Delta t}_t f - U^{-1}_t A U_{j} f = U^{-1}_t [U^{\Delta t}_t A U_{\Delta t} - A] U_t f.$$  
Calling $U_t f = g$, we note that $g$ is in domain $H^n$. But

$$(1/\Delta t)[U^{\Delta t}_t A U_{\Delta t} - A] g$$

$$= (1/\Delta t)(U_{\Delta t} - I)Ag + (U_{\Delta t} A(U_{\Delta t} - I)g)(1/\Delta t).$$

As $\Delta t$ approaches zero, the first term goes to $-iHAg$, since $Ag$ is in domain $H$ by hypothesis. The second term is a little harder to analyze.

First, we note that $A$ defines a closed, and therefore continuous linear
transformation of $B$ into $h$, where $B$ is the Banach space created by giving
domain $H^n$ the graph norm associated with $H^n$.

However $(U_{\Delta t} - I)g/\Delta t$ approaches $iHg$ in $B$ as $\Delta t$ approaches zero,
since $g$ is in domain $H^{n+1}$. Thus $A(U_{\Delta t} - I)g/\Delta t$ converges to $iAHg$ in $h$.

But, finally, from strong continuity of $U_t$ and the fact that $\|U_t\| = 1$
for all $t$, it follows that $U_{-\Delta t} A(U_{\Delta t} - I)g/\Delta t$ approaches $iAHg$ as $\Delta t$
approaches zero.

Collecting what we have proved, we see that $(1/\Delta t)[U_{\Delta t} A U_{\Delta t} - A] g$
approaches $i(AH - HA)g$ as $\Delta t$ approaches zero. From continuity of $U_t$ it
follows that $(1/\Delta t)U_{-\Delta t} (U_{\Delta t} A U_{\Delta t} - A) U_t f$ approaches $i(AH - HA)U_t f$
in $h$ as $\Delta t$ approaches zero. This completes the proof of Theorem 1.

**Corollary 1.** Under the hypotheses of Theorem 1, it cannot happen
that $(i(AH - HA) U_t f, U_t f) > C \|U_t f\|^2$ for any $C > 0$, and neither can it
happen that $(i(AH - HA) U_t f, U_t f) < -C \|U_t f\|^2$ for all $t$ in any infinite
interval.

**Proof.** By the closed graph theorem, and the fact that in the graph
norm generated by $H$ on domain $H$, the norm of $U_t f$ is the same as that
of $f$, it follows that $\|AU_t f\|$ remains bounded. Therefore so does
$(AU_t f, U_t f)$. By Theorem 1, the proof is completed.

**Definition.** An operator $A$ is said to be local with respect to $U_t f$
if $U_t f$ is contained in domain $A$ for all $t$, and $AU_t f$ approaches zero as $t$
approaches $\pm\infty$.
It might at first appear that it is hard to show that an operator $A$ is local with respect to $U_t f$. This is not the case, however, for many types of selfadjoint operators $H$ which are important in applications. A few remarks on this problem seem in order.

First, if $H$ is a selfadjoint operator in $L^2$, and $\mathcal{A}$ is an element of $L^2$, then recall that $\mathcal{A}$ is said to be absolutely continuous with respect to $H$ if the real valued measure $m_\mathcal{A}(S) = \|P(S)f\|^2$ is absolutely continuous with respect to Lebesgue measure on $R$. Here $S$ is any borel set in $R$, and $P(S)$ is the projection associated with $S$ by the spectral measure associated with $H$. The set of all such $\mathcal{A}$ forms a reducing subspace of $H$, and the restriction of $H$ to this subspace forms a selfadjoint operator $H_\mathcal{A}$. $U_t$ takes this subspace into itself. If $H_\mathcal{A} = H$, $H$ is said to be absolutely continuous.

Now, if $\mathcal{A}$ is absolutely continuous with respect to $H$, $H$ is a selfadjoint ordinary differential operator and $\mathcal{A} = L^2(R)$. It follows by an argument in Lax and Phillips [2, p. 147], that $\|C_\mathcal{A} U_t f\|$ approaches 0 as $t$ approaches $\pm \infty$, where $C_\mathcal{A}$ is the characteristic function of any compact interval $\Delta$. Here $H$ must be assumed to have order one or greater. Thus if $\mathcal{A}$ is a bounded operator, and $\mathcal{A}$ is the limit in operator norm of a sequence of operators $A_n$ defined by $A_n f = AC_{\mathcal{A}_n} f$ for a sequence of compact intervals $\Delta_n$, it follows that $\mathcal{A}$ is local with respect to $U_t f$, provided $U_t = e^{iH t}$, $H$ is a selfadjoint ordinary differential operator, and $\mathcal{A}$ is absolutely continuous with respect to $H$. An example of such an $\mathcal{A}$ is multiplication by a $C_0$ function.

It may be shown (see Kato [1]) that many ordinary differential operators have nontrivial absolutely continuous parts, and that therefore such vectors $\mathcal{A}$ may be found. Further, similar considerations can be made to apply to the case where $\mathcal{A}$ is an ordinary differential operator with $C_0$ coefficients, provided $H$ is a selfadjoint ordinary differential operator in $L^2(R)$ with bounded coefficients and nontrivial absolutely continuous part and $\mathcal{A}$ is of order less than or equal to that of $H$.

Another way of showing locality, which also applies to differential operators is contained in the following theorem.

**Theorem 2.** Let $H$ be absolutely continuous. Let $\mathcal{A}$ be $H$-compact. Then $\mathcal{A}$ is local with respect to $U_t f$ for all $f$ in domain $H$.

**Remark.** Recall that $\mathcal{A}$ is said to be $H$-compact if domain $\mathcal{A}$ contains domain $H$, and $\mathcal{A}$ is a compact operator from domain $H$ into $h$, where domain $H$ is equipped with the graph norm from $H$.

**Proof.** Since $H$ is absolutely continuous, then by the Riemann-Lebesgue lemma $(U_t f, g)$, which equals $(\int_{-\infty}^{\infty} e^{iH t} dP f, g)$, approaches 0 when $t$ approaches $\pm \infty$ for all $f$ and $g$ in $h$. Now suppose $\mathcal{A}$ is not local.
with respect to some \( f \) in domain \( H \). Then there is a sequence \( t_n \) approaching, say, \( +\infty \) such that \( \|AU_{t_n}f\| > C \) for some \( C > 0 \). Since \( A \) is \( H \)-compact, it follows that, for some subsequence \( t_{n(i)} \), \( AU_{t_{n(i)}}f \) approaches \( g \), with \( \|g\| \geq C \).

Let \( U_{t_{n(i)}}f = f_i \). Let \( \|Af_i\| \leq M \). Select \( g_1 \) in domain \( A^* \) such that \( \|g_1 - g\| \leq C^2 / 2M \). Then \( \|(Af_i, g_1) - (Af_i, g)\| \leq M \|g_1 - g\| \leq C^2 / 2 \). Therefore, when \( j \) is large enough, \( \|(Af_j, A^*g_1)\| \leq C^2 / 4 \). Therefore \( \|(Af_j, A^*g_1)\| \leq C^2 / 4 \), which contradicts the fact \( f_j \) converges weakly to 0. This completes the proof.

We now use the hypothesis of locality with respect to \( U_{t_1}f \).

**Theorem 3.** Suppose that \( AH^2 \), \( HAH \) and \( H^2A \) are all defined on domain \( H^{n-1} \) for some positive integer \( n \geq 3 \). Suppose \( A \) is local with respect to \( U_{t_1}f \), where \( f \) is in domain \( H^n \), and suppose there is a dense subspace \( S \) of \( h \) on which \( A^*H^2 \), \( HA^*H \) and \( H^2A^* \) are defined. Then \( AH - HA \) is local with respect to \( U_{t_1}f \).

**Remark.** If \( A \) is symmetric, the last hypothesis is obviously redundant. Also if \( A \) and \( H \) are ordinary differential operators, \( C^2 \) will usually be such a subspace \( S \).

**Proof.** \((U_{t_1}^{-1}AU_{t_1}f)^* = U_{t_1}^{-1}(AH^2 - 2HAH + H^2A)U_{t_1}f \) as may be seen using Theorem 1. It is of course necessary to show that the operator \( AH - HA \), when restricted to domain \( H^{n-1} \), has a closed extension. However, an operator has a closed extension if and only if its adjoint is densely defined, so our last hypothesis takes care of this possibility.

Now if \( T \) is the operator \( AH^2 - 2HAH + H^2A \), restricted to domain \( H^n \), then \( T \) has a closed extension. Therefore, giving domain \( H^n \) the graph norm from \( H^n \), and letting \( \hat{T} \) denote the operator induced by \( T \) from \( H^n \) to \( H^n \), we see that \( \hat{T} \) is continuous by the closed graph theorem.

But since \( H^nU_{t_1}f = U_{t_1}H^n f \), it follows that all \( U_{t_1}f \) have the same norm as \( f \) in the graph norm on domain \( H^n \). Therefore the set of all \( U_{t_1}f \) is a bounded set in the Banach space domain \( H^n \), so that the set of \( TU_{t_1}f \) is a bounded set in \( h \). Therefore the set of all \( (U_{t_1}^{-1}AU_{t_1}f)^* \) is a bounded set in \( h \), as \( t \) ranges over the whole real line.

Let \( f_t \) be \( U_{t_1}^{-1}AU_{t_1}f \). We need to show that \( f_t \) approaches zero in norm, as \( t \) approaches \( \pm \infty \), in order to prove the theorem. Let \( g(t) = (f_t, f_t) \).

Then \( g'(t) = (f'_{t}, f_t) + (f_t, f'_{t}) \). Also, \( g''(t) = (f''_{t}, f_t) + 2(f'_t, f'_t) + (f_t, f''_{t}) \). Since \( f''_{t} \) is bounded, and \( f_t \) goes to zero as \( t \) approaches \( \pm \infty \), it follows that \( (f''_{t}, f_t) \) also approaches zero.

To show that \( (f''_{t}, f_t) \) approaches 0, we first observe that

\[ f_t = iU_{t_1}^{-1}(AH - HA)U_{t_1}f. \]
which, once again, by the closed graph theorem, remains bounded as $t$ approaches $\pm \infty$. Thus $(f'_n, f'_n)$, which equals $(f'_n, f''_n) + (f'_n, f''_n)$, is a bounded real valued function of $t$. If there were a sequence $t_n$ approaching, say, $+\infty$ such that $(f'_n, f'_n) > \varepsilon$, then there would have to be a $\delta$ such that $(f'_n, f'_n)$ remained $\geq \varepsilon/2$ on $[t_n - \delta, t_n + \delta]$ for all $n$, by the mean value theorem. Thus there would be an $N$ such that $g''$ would be greater than $\varepsilon/4$ on the interval $[t_n - \delta, t_n + \delta]$ for all $n \geq N$. However, since $f'_n$ approaches zero and $f'_n$ remains bounded as $t$ approaches infinity, it is clear that $g'(t)$ must approach 0. This contradicts the fact we just discovered about $g''$. The theorem is proved.

**Corollary 2.** Under the hypotheses of Theorem 3, it cannot happen that $AH - HA$ has a bounded inverse when restricted to the linear span of the $U_tf$.

**Corollary 3.** Let $f$ be as in Theorem 3, and suppose $f$ is perpendicular to the eigenvectors of $H$. Let $T$ be the operator formed by restricting $AH - HA$ to the linear span of the $U_tf$, and $T_1$ be the closure of the graph of $T$ in the product space $h \times h$. Then $T_1$ cannot be a linear operator with closed range and finite dimensional null space.

**Proof.** There is a sequence $t_n$ approaching infinity such that $U_tf$ approaches 0 weakly in $h$. (See Lax and Phillips [2, p. 145].) But $(AH - HA)U_tf$ approaches zero by Theorem 3.

Let $S$ be the closed linear span of the $U_tf$. If $T_1$ is a closed operator defined on a dense subspace of $S$, and $K$ is its null space, and $T_1$ has closed range, then by dividing out $K$ and using the closed graph theorem we see that the distance from $U_tf$ to $K$ approaches zero. From this fact, and the fact that $K$ is finite dimensional, it follows that a subsequence of $U_tf$ converges to a point $g$ of $K$, with $\|g\| = \|f\|$. This contradicts the weak convergence of $U_tf$ to zero.

**Corollary 4.** Let $H$ be absolutely continuous, and $A$ be $H$-compact and symmetric. Further, suppose that for some positive integer $n$, $AH^n$, $HAH$ and $H^nA$ are defined on domain $H^n$. Then the restriction of $AH - HA$ to domain $H^n$ can have no extension to a closed operator in $h$ with closed range and finite dimensional null space.

**Proof.** Combine Theorem 2 and Corollary 3.

**References**


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