CAUCHY SEQUENCES IN SEMIMETRIC SPACES

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Abstract. As the main result we prove that every semimetris-
able space has a semimetric for which every convergent sequence
has a Cauchy subsequence. This result is used to show that a $T_1$
space $X$ is semimetrizable if and only if it is a pseudo-open
π-image of a metric space.

In a metric space it is easy to show that every convergent sequence is a
Cauchy sequence; however, there are numerous examples of semimetric
spaces in which there exist convergent sequences with no Cauchy sub-
sequence. The main purpose of this note is to show that every semi-
metrizable space has a compatible semimetric for which every convergent
sequence has a Cauchy subsequence.

Unless otherwise stated no separation axioms are assumed, and we let
$N$ denote the set of natural numbers.

Let $X$ be a topological space and $d$ a real valued, nonnegative symmetric
function defined on $X \times X$ such that $d(x, y)=0$ if and only if $x=y$. The
function $d$ is called a semimetric for $X$ provided: For $A \subseteq X$, $x \in A$
if and only if $d(x, A)=\inf\{d(x, a): a \in A\}=0$. The function $d$ is called a symmetric
[3] for $X$ provided: A set $A \subseteq X$ is closed if and only if $d(x, A)>0$ for any
$x \in X-A$. See [4] for a discussion of the differences between a semimetric
space and a symmetric space.

If $(X, d)$ is a semimetric space a sequence $\{x_n\}_{n=1}^\infty$ in $X$ is said to be
Cauchy if for any $\varepsilon>0$ there is some $k \in N$ such that $d(x_n, x_m)<\varepsilon$ for all
$n, m \geq k$. See [8] for the definition of weak completeness in semimetric
spaces which can be stated in terms of Cauchy sequences.

Example. Let $X=A \cup B$ where $A=\{(0, y) \in \mathbb{R}^2: -1 \leq y \leq 1\}$ and $B$ is the
graph of the equation $y=\sin(1/x)$ for $0<x \leq 1$. If $u=(x_1, y_1)$ and $v=
(x_2, y_2)$ are elements of $X$ define $d(u, v)=[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$ if at least
one of $u$ or $v$ is in $A$. Define $d(u, v)$ to be the “arc length” between $u$ and $v$
if both $u$ and $v$ are in $B$. Then $d$ is a semimetric for the usual topology on $X$
and there are no Cauchy sequences in $B$ which converge to points in $A$.

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Theorem 1. If $(X, d)$ is any semimetric space (or symmetric space) the following conditions are equivalent:

(a) Every convergent sequence in $X$ has a Cauchy subsequence.

(b) If $(x_n)_{n=1}^\infty$ is a convergent sequence in $X$ and $\varepsilon$ is a positive number there is a subsequence $(z_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $d(z_n, z_m) < \varepsilon$ for all $n, m \in \mathbb{N}$.

(c) If $F \subseteq X$ and there is a positive $\varepsilon$ such that $d(x, y) \geq \varepsilon$ for all distinct $x, y \in F$ then $F$ is closed.

Proof. The proofs that (b) $\implies$ (a) and (a) $\implies$ (c) are relatively easy and are left to the reader.

To show that (c) $\implies$ (b) we assume (c) is true and let $(x_n)_{n=1}^\infty$ be a sequence in $X$ which converges to $x \in X$. We may assume $x \neq x_n$ for any $n \in \mathbb{N}$. Let $\varepsilon > 0$. Suppose that for every subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ there is a subsequence $(z_n)_{n=1}^\infty$ of $(y_n)_{n=1}^\infty$ such that $d(z_1, z_n) \geq \varepsilon$ for all $n > 1$. Then we could construct a subsequence $(z'_n)_{n=1}^\infty$ such that $d(z'_n, z'_m) \geq \varepsilon$ for all distinct $n, m \in \mathbb{N}$. This is impossible; so there exists a subsequence of $(x_n)_{n=1}^\infty$ such that for every subsequence $(z_n)_{n=1}^\infty$ we have $d(z_n, z_m) < \varepsilon$ for some $n > 1$. It follows that we can find a subsequence $(z'_n)_{n=1}^\infty$ of $(y_n)_{n=1}^\infty$ such that $d(z'_n, z'_m) < \varepsilon$ for all $n, m \in \mathbb{N}$. That completes the proof of the theorem.

Theorem 1 is useful in that it illustrates the effect that condition (a) has on the topology on $X$ as in condition (c). In Theorem 2 it turns out that condition (c) is much easier to use than condition (a). Condition (c) has been discussed previously in several papers (see [2], [7], [9]), and is known as the weak condition of Cauchy. In [7] there is an example of a symmetric space which has no symmetric satisfying the weak condition of Cauchy. Hence the analogue of Theorem 2 for symmetric spaces is not true.

If $d$ is a semimetric for $X$ and $x \in X$ we let $S_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ denote the sphere about $x$ of radius $\varepsilon$. It is easy to show that if $d$ is a semimetric for which the spheres are open sets then every convergent sequence in $X$ has a Cauchy subsequence. In [6] however, Heath has given an example of a regular semimetric space for which there is no semimetric with respect to which all spheres are open.

To further illustrate that Theorem 2 is about as strong as could be expected we point out that it is known (see [5]) that a $T_1$ space $X$ is developable if and only if $X$ is semimetrizable by a semimetric under which all convergent sequences are Cauchy.

Theorem 2. Every semimetric space $(X, d)$ is semimetrizable by a compatible semimetric $\rho$ where every convergent sequence in $X$ has a Cauchy subsequence.
Proof. For any nonnegative real number \( r \) we define

\[
\mathcal{A}(r) = \{ B \subseteq X : d(x, y) \geq r \text{ for all } x, y \in B, x \neq y \}.
\]

Let \( x, y \) be arbitrary elements of \( X \) and define

\[
A(x, y) = \{ z : \text{there exists } B \in \mathcal{A}(d(x, y)/2), \text{ with } z \in B \text{ and } x, y \in B \}.
\]

Now define \( \rho(x, y) = \inf\{ d(x, z) + d(z, y) : z \in A(x, y) \} \).

It is clear that \( \rho(x, y) = \rho(y, x) \) and \( \rho(x, y) \geq 0 \) for any \( x, y \in X \). We also note that \( \rho(x, y) \leq d(x, y) \), so

\[
S_\rho(x, \varepsilon) = \{ y \in X : d(x, y) < \varepsilon \} \subseteq S_d(x, \varepsilon) = \{ y \in X : \rho(x, y) < \varepsilon \}
\]

for \( x \in X \) and \( \varepsilon > 0 \).

Next we show that for \( x \in X \) and \( \varepsilon > 0 \) there exists an integer \( m \) such that \( S_\rho(x, 1/m) \subseteq S_d(x, \varepsilon) \). From this it follows that \( \rho(x, y) > 0 \) if \( x \neq y \) and \( \rho \) is a semimetric for the topology induced by \( d \).

Assume otherwise and suppose there exists \( y_n \in S_\rho(x, 1/n) - S_d(x, \varepsilon) \) for each \( n \in \mathbb{N} \). Consider an arbitrary \( n \in \mathbb{N} \) such that \( 1/n < \varepsilon \). Then \( d(x, y_n) = \varepsilon_n \geq \varepsilon \) and \( \rho(x, y_n) = 1/n < \varepsilon \) implies that there is a \( z_n \in A(x, y_n) \) such that \( d(x, z_n) + d(z_n, y_n) < 1/n \). Hence \( d(x, z_n) < 1/n \). Since \( z_n \in A(x, y_n) \), there is \( B_n \in \mathcal{A}(\varepsilon_n/2) \) such that \( x, y_n \in B_n \) and \( z_n \in B_n \). Let \( \{ z_{n,i} \}_{i=1}^{\infty} \) be a sequence in \( B_n \) such that \( z_{n,i} \rightarrow z_{n} \). Now \( d(x, z_{n,i}) < 1/n \), so \( z_{n,i} \rightarrow x \) and there exists \( m \in \mathbb{N} \) such that \( 1/m < \varepsilon/2 \) and \( z_{n,i} \in S_\rho(x, \varepsilon/2) \). Hence \( \text{int}(S_\rho(x, \varepsilon/2)) \) is an open set about \( z_{n,i} \) and there must exist \( i \in \mathbb{N} \) such that \( z_{n,i} \in \text{int}(S_d(x, \varepsilon/2)) \). This implies \( d(x, z_{n,i}) < \varepsilon/2 \). But \( z_{n,i} \in B_m \in \mathcal{A}(\varepsilon_m/2) \) and \( x \in B_m \), which implies \( d(x, z_{n,i}) \geq \varepsilon_m/2 \). This is a contradiction since \( \varepsilon_m \geq \varepsilon \). Hence there exists \( n \) such that \( S_\rho(x, 1/n) \subseteq S_d(x, \varepsilon) \) and it follows that \( \rho \) is a semimetric compatible with the topology on \( X \).

To complete the proof of the theorem we show that \( (X, \rho) \) satisfies condition (c) of Theorem 1. Suppose \( F \) is a subset of \( X \) and \( \varepsilon > 0 \) is a positive number such that \( \rho(x, y) \geq \varepsilon \) when \( x, y \in F \), \( x \neq y \). If \( F \) is not closed let \( z \in F - F \) and let \( B = F \cap S_d(z, \varepsilon/2) \). Let \( \varepsilon_1 = \inf\{ d(x, y) : x, y \in B \} \); then \( \varepsilon_1 \geq \varepsilon > 0 \) and \( B \in \mathcal{A}(\varepsilon_1) \). There must be \( x, y \in B \) such that \( \varepsilon_1 \leq d(x, y) < 2\varepsilon_1 \). Hence \( d(x, y)/2 < \varepsilon_1 \) and it follows that \( B \in \mathcal{A}(d(x, y)/2) \). Also \( z \in B \), so \( z \in A(x, y) \). Hence \( \rho(x, y) \leq d(x, z) + d(z, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon \). This is impossible since \( x, y \in F \) and \( \rho(x, y) \geq \varepsilon \). So \( F \) is closed.

It is worthwhile noting that if \( (X, d) \) is a semimetric space which already satisfies condition (a) of Theorem 1 and if \( \rho \) is the semimetric as constructed in the proof of Theorem 2 then \( \rho = d \).

As an application of Theorem 2 we have the corollary below. See [3] for the definition of a \( \pi \)-mapping and a pseudo-open mapping as well as a discussion of related results on characterizing certain topological spaces as images of metric spaces.
Corollary 3. A $T_1$ space $X$ is semimetrizable if and only if it is a pseudo-open $\pi$-image of a metric space.

Proof. Alexander [1] has shown that a $T_1$ space $X$ is a pseudo-open $\pi$-image of a metric space if and only if it is semimetrizable with a semi-metric satisfying (a) of Theorem 1. Apply Theorem 2 and the proof is complete.

References