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ON TORSION IN H-SPACES OF RANK TWO

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ABSTRACT. Let X be an arcwise connected H-space that is also a finite CW complex. If X has rank two and if its mod 2 cohomology is primitively generated, then X has no p-torsion for $p \geqq 5$.

Let X be an arcwise connected H-space that is also a finite CW complex. A classical theorem of Hopf says that the reduced cohomology of X with rational coefficients is an exterior algebra on odd dimensional generators. If it is generated by $x_1, \cdots, x_n$ in dimensions $d_1, \cdots, d_n$, respectively, with $d_i \leqq d_{i+1}$, then X is said to have *rank* n and have *type* $(d_1, \cdots, d_n)$. If $H^i(X; Z)$ contains an element of order p for some i and some prime p, then X is said to have *p-torsion*.

In §6 of **[8]** it is shown that if X is an arcwise connected H-space that is also a finite CW complex and if X is of rank one, then X has no p-torsion for all odd primes p. The purpose of this note is to show:

THEOREM. *Let X be an arcwise connected H-space that is also a finite* CW *complex. If X has rank two and if its* mod 2 *cohomology is primitively generated, then X has no p-torsion for* $p \geqq 5$.

PROOF. We divide into two cases, namely: (i) X has no 2-torsion, and (ii) X has 2-torsion.

(i) If X has no 2-torsion, then the type of X is (1, 1), (1, 3), (1, 7), (3, 3), (3, 5), (3, 7), or (7, 7). We remark here that this result was partially proved by J. Adams **[3]** and completed by Douglas-Sigrist **[9]** and was also proved independently by J. Hubbuck **[10**, Theorem 1.1]. Suppose that X has p-torsion for $p \geqq 5$. If s is the smallest integer for which $H^s(X; Z)$ has p-torsion, then, by [7, Theorem 6.4], $s=2m$ and, since $p \neq 2$, $H_{2m}(X; Z_p)$ has a primitive element. Since $P(H_{2m}(X; Z_p)) \cong (Q(H^{2m}(X; Z_p))^*$ **[11]**, where $P(H_{2m}(X; Z_p))$ is the subspace of primitive elements of $H_{2m}(X; Z_p)$, and $(Q(H^{2m}(X; Z_p))^*$ is the dual of the subspace of indecomposable elements of $H^{2m}(X; Z_p)$, we see that $H^{2m}(X; Z_p)$ has an indecomposable element; hence $H^{2m}(X; Z_p)$ has a generator. By [7, Theorem 4.7], $H^*(X; Q)$ has a generator in dimension $2mp^k - 1$, for some k, $0 < k < \infty$.

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We have $2m \geqq 2$, $p \geqq 5$, and $k \geqq 1$; so $2mp^k - 1 \geqq 9$. This contradicts the possible types of X which are (1, 1), (1, 3), (1, 7), (3, 3), (3, 5), (3, 7) and (7, 7).

(ii) If X has 2-torsion, then again we consider two cases: (a) X is simply connected, and (b) X is not simply connected.

(a) If X has 2-torsion and is simply connected, then we have:

$$H^*(X; Z_2) \cong H^*(G_2; Z_2),$$

where the exceptional Lie group G_2 has mod 2 cohomology [**4**]:

$$H^*(G_2; Z_2) \cong Z_2[x_3]/(x_3^4) \otimes \bigwedge (Sq^2 x_3),$$

the tensor product of a polynomial algebra on one generator of dimension 3 truncated at height 4 and an exterior algebra on one generator of dimension 5. Since $H^{14}(G_2; Z_2) \neq 0$ and $H^i(G_2; Z_2) = 0$ for $i > 14$, we have $H^{14}(X; Z_2) \neq 0$ and $H^i(X; Z_2) = 0$ for $i > 14$. By [**6**, Theorem 7.1] we have $H^{14}(X; Q) \neq 0$ and $H^i(X; Q) = 0$ for $i > 14$. Now if X has p-torsion for $p \geqq 5$, then by the argument used in (i) above we see that there is a generator for $H^*(X; Q)$ in dimension $2mp^k - 1$, where $2m$ is the smallest integer for which $H^{2m}(X; Z)$ has p-torsion. Since X is simply connected, we have $\pi_1(X) = \pi_2(X) = 0$ by [**6**, Theorem 6.11]. This implies that $H^1(X) = H^2(X) = 0$. Thus $2m \geqq 4$, $p \geqq 5$, $k \geqq 1$, and so $2mp^k - 1 \geqq 19$. This contradicts the fact that $H^i(X; Q) = 0$ for $i > 14$.

(b) If X has 2-torsion and is not simply connected, then consider the universal covering space $\tilde{X}$ of X. First, observe that X does not have type (1, 1). If X has type (1, 1), then $H^2(X; Z) = Z$ and $H^i(X; Z) = 0$ for $i > 2$ [**6**, Corollary 7.2]. Since X has 2-torsion, $H^{2n}(X; Z)$ contains an element of order 2, where $2n$ is the smallest integer for which $H^{2n}(X; Z)$ has 2-torsion. Since $2n \geqq 2$, we have a contradiction; hence X does not have type (1, 1). Also, notice that $\tilde{X}$ is a simply connected H-space and that X is of the homotopy type of a finite CW complex [**12**]. From [**5**] we have that if $(d_1, \cdots, d_n)$ is the type of $\tilde{X}$, then the possible types of X are $(d_1, \cdots, d_n)$ and $(1, 1, \cdots, 1, d_1, \cdots, d_n)$. Let $\tilde{X}$ have type (d_1), then by [**1**] or [**8**] we have $d_1 = 3$ or 7. This implies that X has type (1, 3) or (1, 7), in which case there is no p-torsion for $p \geqq 5$ by the argument used in (i) above. Now, let $\tilde{X}$ have rank two. From (i) above, we see that if $\tilde{X}$ has no 2-torsion, then $\tilde{X}$ has type (3, 3), (3, 5), (3, 7), or (7, 7). So the type of X is (3, 3), (3, 5), (3, 7), or (7, 7). As reasoned in (i) above, X has no p-torsion for $p \geqq 5$ when $\tilde{X}$ has no 2-torsion. If $\tilde{X}$ has 2-torsion, then again we have:

$$H^*(\tilde{X}; Z_2) \cong H^*(G_2; Z_2).$$

X has no p-torsion for $p \geqq 7$ because if it does, $H^*(X; Q)$ will have a generator in a dimension at least $2(1)7 - 1 = 13$. This implies that the type of X is $(1,13)$ since the top dimension of $H^*(X;Q)$ is 14. But then $\tilde{X}$ is of rank one, a contradiction to our hypothesis that $\tilde{X}$ is of rank two. Suppose X has 5-torsion. Then $H^*(X;Q)$ has a generator in dimension $2m5^k-1$ for $m\geqq 1$, $k\geqq 1$, where $2m$ is the smallest integer for which $H^{2m}(X;Z)$ has 5-torsion. The only dimension to consider is 9 since the top dimension is 14. Again this implies that the type of X is $(5, 9)$. Now, X has 2-torsion, so let $2n$ be the smallest dimension in which $H^*(X;Z)$ has 2-torsion. We see that $2n$ is not 2, 4, or 8 since then $H^*(X;Q)$ will have a generator in dimension $2(1)2^k-1$, $2(2)2^k-1$, or $2(4)2^k-1$, respectively, which is not 5 or 9. Thus 6 and 10 are the only remaining possibilities. In either case $H^3(X;Z_2)=H^1(X;Z_2)=0$. But $H^3(\tilde{X};Z_2)\simeq H^3(G_2;Z_2)\simeq Z_2$ so by **[5]** we have a contradiction. This completes the proof of the theorem.

REMARK. The theorem above says that X has no p-torsion for $p\geqq 5$. In fact, X may have 2-torsion; simply let $X=G_2$. Also, X may have 3-torsion. Consider $SU(3)$. Its center is Z_3 **[2]**. Let X be the corresponding projective group $PSU(3)$. It is obvious that $H^2(X;Z)\simeq Z_3$.

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