ON A CONVERSE OF YOUNG'S INEQUALITY

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Abstract. A converse of Young's inequality is proved through the formulation of a functional inequality.

The converse of Young's inequality presented in this note differs from the converses given in Mitrinović [2] and Takahashi [3].

Theorem. Let $F$ denote the collection of all real-valued functions $f$ defined for all $0 \leq t < \infty$ such that

(i) $f(0) = 0$,
(ii) $f$ is continuous for $0 \leq t < \infty$ (from the right at $t=0$),
(iii) $f$ is strictly increasing for $0 \leq t < \infty$, and $\lim_{t \to \infty} f(t) = +\infty$.

Suppose that $T : F \to F$ is an operator such that

(1) $T[f](0) = 0$, $f \in F$,
(2) $xy = T[f](x) + T[f^{-1}](y)$, $f \in F, x \geq 0, y \geq 0$,
(3) $\forall f \in F, ab = T[f](a) + T[f^{-1}](b)$, if $b = f(a)$.

Then

$$T[f](x) = \int_0^x f(t) \, dt, \quad x \geq 0.$$  

Proof. From (3), it follows that

(4) $T[f^{-1}](f(x)) = xf(x) - T[f](x)$ and
(5) $T[f](f^{-1}(x)) = xf^{-1}(x) - T[f^{-1}](x)$ since $f(f^{-1}(x)) = x$.

In (2), replace $y$ by $f(x)$, $x$ by $x+h$, then use (4) to infer that

$$(x+h)f(x) \leq T[f](x+h) + T[f^{-1}](f(x)) = T[f](x+h) + xf(x) - T[f](x).$$

This implies that

(6) $hf(x) = (x+h)f(x) - xf(x) \leq T[f](x+h) - T[f](x)$

holds for $x > 0$ and small $h$.  

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With $f^{-1}(t) = x + h$, equality (5) leads to
\[ T[f](x + h) - T[f](x) = T[f][f^{-1}(t)] - T[f](x) \]
\[ = tf^{-1}(t) - T[f^{-1}](t) - T[f](x) \]
\[ = (x + h)f(x + h) - T[f^{-1}](t) - T[f](x) \]
\[ = hf(x + h) + xf(x + h) - T[f](x) - T[f^{-1}](x) \]
\[ = hf(x + h), \]

since $xf(x + h) - T[f](x) \leq T[f^{-1}](f(x + h))$ by (2). Combine (6) and (7) to infer that
\[ f(x) \leq \frac{T[f](x + h) - T[f](x)}{h} \leq f(x + h) \]

holds for $x \geq 0$ and for small positive $h$. Clearly the reverse inequalities hold for $x > 0$ and for small negative $h$. The continuity of $f$ implies that $dT[f](x)/dx$ exists and equals $f(x)$ for $x \geq 0$. Thus $T[f](x) = T[f](x) - T[f](0) = \int_0^x f(t) \ dt$. This completes the proof.

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REFERENCES


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