ON A CONVERSE OF YOUNG’S INEQUALITY

IH-CHING HSU

Abstract. A converse of Young’s inequality is proved through the formulation of a functional inequality.

The converse of Young’s inequality presented in this note differs from the converses given in Mitrinović [2] and Takahashi [3].

Theorem. Let $F$ denote the collection of all real-valued functions $f$ defined for all $0 \leq t < \infty$ such that

(i) $f(0) = 0,$
(ii) $f$ is continuous for $0 \leq t < \infty$ (from the right at $t=0$),
(iii) $f$ is strictly increasing for $0 \leq t < \infty$, and $\lim_{t \to \infty} f(t) = +\infty.$

Suppose that $T: F \to F$ is an operator such that

(1) $T[f](0) = 0,$ $f \in F,$
(2) $xy \leq T[f](x) + T[f^{-1}](y),$ $f \in F,$ $x \geq 0,$ $y \geq 0,$
(3) $\forall f \in F, \ ab = T[f](a) + T[f^{-1}](b),$ if $b = f(a)$.

Then

$$T[f](x) = \int_0^x f(t) \, dt, \quad x \geq 0.$$ 

Proof. From (3), it follows that

(4) $T[f^{-1}](f(x)) = xf(x) - T[f](x)$ and
(5) $T[f](f^{-1}(x)) = xf^{-1}(x) - T[f^{-1}](x)$ since $f(f^{-1}(x)) = x.$

In (2), replace $y$ by $f(x),$ $x$ by $x+h,$ then use (4) to infer that

$$(x + h)f(x) \leq T[f](x + h) + T[f^{-1}](f(x))$$

$$= T[f](x + h) + xf(x) - T[f](x).$$

This implies that

(6) $hf(x) = (x + h)f(x) - xf(x) \leq T[f](x + h) - T[f](x)$ holds for $x > 0$ and small $h.$

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With \( f^{-1}(t) = x + h \), equality (5) leads to

\[
T[f](x + h) - T[f](x) = T[f][f^{-1}(t)] - T[f](x)
= tf^{-1}(t) - T[f^{-1}](t) - T[f](x)
= (x + h)f(x + h) - T[f^{-1}](t) - T[f](x)
= hf(x + h) + xf(x + h) - T[f](x) - T[f^{-1}](t)
\leq hf(x + h) + T[f^{-1}][f(x + h)] - T[f^{-1}](t)
= hf(x + h),
\]

since \( xf(x + h) - T[f](x) \leq T[f^{-1}][f(x + h)] \) by (2). Combine (6) and (7) to infer that

\[
f(x) \leq \frac{T[f](x + h) - T[f](x)}{h} \leq f(x + h)
\]

holds for \( x \geq 0 \) and for small positive \( h \). Clearly the reverse inequalities hold for \( x > 0 \) and for small negative \( h \). The continuity of \( f \) implies that \( dT[f](x)/dx \) exists and equals \( f(x) \) for \( x \geq 0 \). Thus \( T[f](x) = T[f](x) - T[f](0) = \int_0^x f(t) dt \). This completes the proof.

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**References**


Department of Mathematics, Northern Illinois University, DeKalb, Illinois 60115