PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

SPACES WITH GIVEN HOMEOMORPHISM GROUPS

M. C. THORNTON

ABSTRACT. A topological space X is constructed so that its group of homeomorphisms is isomorphic to a given finitely generated group G. If G has r generators and finite order n, X has $n(2r+1)$ points. The relation of X to certain covering spaces is considered.

1. **Introduction.** Let G be a finitely generated group presented with r generators and relators $\{R_\alpha\}$. A topological space X is constructed so its group of homeomorphisms is isomorphic to G. X is a T_0 space with the property that every point has a minimal open set containing it. Such spaces, called A-spaces, have been studied in [**6**] and [**7**]. If G has finite order n, X will have $n(2r+1)$ points. A space X_r is constructed which is of the weak homotopy type of a wedge of r circles. X is the same homotopy type as the covering space of X_r corresponding to the normal subgroup generated by the relators $\{R_\alpha\}$.

The general problem of realizing a group as the total automorphism group of some mathematical object goes back almost one hundred years to Cayley and the idea of a Cayley diagram or group graph. There is a one-one correspondence between T_0 A-spaces and partially ordered sets [**7**, p. 327]. There is clearly a similar correspondence between these partially ordered sets and directed graphs with no loops. Both these correspondences preserve automorphisms. Thus graph automorphisms have a direct bearing on T_0 A-space homeomorphisms. A great deal is known about automorphism groups of graphs. A good source with excellent references is [**5**, Chapter 14].

Let G have order n with r generators. Cayley showed in 1878 that G may be represented as the automorphism group of an r-colored, directed graph with n points. In 1938 Frucht [**2**] showed that, dropping the colored and directed conditions, a graph of $n(2r+1)(r+1)$ points could be used. Later he showed [**3**] that for groups of order at least three a regular degree 3 graph of $n(2r+4)$ points could be used and if the group was not cyclic, $n(2r)$ points were enough. In 1945 Birkhoff [**1**] proved that G could be

Presented to the Society, March 8, 1971; received by the editors March 11, 1971.

AMS 1970 *subject classifications*. Primary 54H15; Secondary 54B99, 20F05.

Key words and phrases. Finite topological space, A-space, homeomorphism groups, transformation groups, covering spaces, covering transformations.

represented as the automorphisms of a partially ordered set with $n(n+1)$ elements.

Using a Cayley diagram, de Groot [4] constructed an infinite directed graph representing any group G (not assumed finite) as a group of automorphisms. He modified this construction to obtain a compact connected Hausdorff space with G as homeomorphism group. Since his infinite directed graph had no loops G could be viewed as the homeomorphisms of an infinite T_0 A-space.

The construction described here gives a partially ordered set of $n(2r+1)$ elements improving slightly Birkhoff's result, gives a finite T_0 A-space for finite groups improving de Groot's result, and gives a smaller graph than Frucht constructed, but at the expense of assuming directedness.

The author wishes to thank G. Birkhoff, P. Doyle, and the referee for helpful comments on the references.

2. **Example.** We first give the construction for a specific group. Let G be the dihedral group with generators h, k and relations $h^2=e$, $k^3=e$, $hkh=k^2$. For each $g \in G=\{e, h, k, k^2, hk, hk^2\}$ we form a subspace $X(g)$ with points $L(g)$, $S(g)$, $T(g)$, $A(g)$, $B(g)$, $L(gh)$, and $L(gk)$. In Figure 1, the

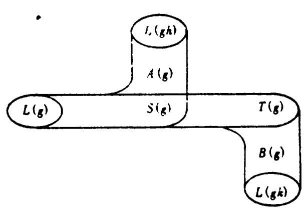

FIGURE 1. $X(g)$ where G has 2 generators.

simple closed curves indicate the minimal open sets which are a base for the topology of $X(g)$. For example, the smallest open set containing $A(g)$ is $\{L(g), S(g), A(g), L(gh)\}$. X is defined to be the union of the $X(g), g \in G$, where we identify $L(gh)$ in $X(g)$ with $L(gh)$ in $X(gh)$ and $L(gk)$ in $X(g)$ with $L(gk)$ in $X(gk)$.

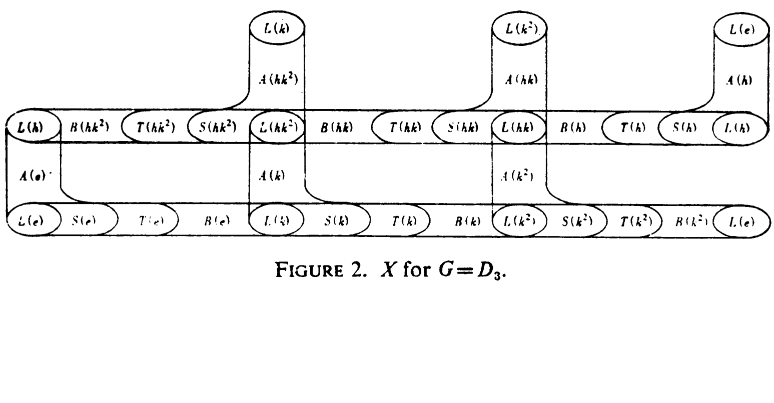

FIGURE 2. X for $G=D_3$.

The resulting space is given in Figure 2 where similarly labeled L points are to be identified. For a point $Y(x)$ in X we shall call $Y=L, S, T, A$, or B the type of the point and $x \in G$ the label of the point.

In order to prove that the group of homeomorphisms of X is isomorphic to G we show for each $g \in G$ there is a unique homeomorphism $\phi(g)$ such that $\phi(g)(L(e))=L(g)$ and $\phi(g)\phi(g')=\phi(gg')$. Define the function $\phi(g)$ by $\phi(g)(Y(x))=Y(gx)$. Clearly $\phi(g)\phi(g')=\phi(gg')$ and $\phi(g)^{-1}=\phi(g^{-1})$. To see that $\phi(g)$ is continuous, we check the inverse image of the minimal open sets. Now $\phi(g)^{-1}\{L(x)\}=L(g^{-1}x)$ is open, $\phi(g)^{-1}\{L(x), S(x)\}=\{L(g^{-1}x), S(g^{-1}x)\}$ is open, etc. In fact the inverse image of any minimal open set of a point is a minimal open set of another point of the same type where all the labels of the points have been left multiplied by g^{-1}. Since $\phi(g)^{-1}=\phi(g^{-1})$ is likewise continuous, $\phi(g)$ is a homeomorphism.

Now we show that a homeomorphism $\psi: X \rightarrow X$ is completely determined by the image of one point. Two points have homeomorphic minimal open sets if and only if they have the same type. Therefore ψ preserves the type of a point and may change the label. Suppose $\psi(A(u))=A(v)$. Then since minimal open sets must be preserved, $\psi(S(u))=S(v)$ and $\psi(L(u))=L(v)$. This forces $\psi(L(uh))=L(vh)$. Similarly we may see the image of any point in a minimal open set uniquely determines ψ on that minimal open set. Since h and k generate G any two L points can be joined by a chain of overlapping minimal open sets of points of type A or B. For suppose $L(v)$ and $L(w)$ are given and $v^{-1}w=f_1 f_2 \cdots f_n$ where each f_i is either h, k, or their inverses. Then if $f_1=h$ or h^{-1}, the minimal open set of $A(v)$ or $A(vh^{-1})$ contains $L(v)$ and $L(vf_1)$. If $f_2=k$ or k^{-1} then the minimal open set of $A(vf_1)$ or $A(vf_1k^{-1})$ contains $L(vf_1)$ and $L(vf_1 f_2)$. Continuing this process we obtain a chain of minimal open sets from $L(v)$ to $L(vf_1 f_2 \cdots f_n)=L(w)$. Thus the ψ image of any L point determines the image of all other L points and hence the image of all points. Therefore if $\psi(L(e))=L(g)$ we have $\psi=\phi(g)$. This shows that the map $g \rightarrow \phi(g)$ is an isomorphism of G onto the group of homeomorphisms of X.

3. **Construction.** For an arbitrary finitely generated group G, the construction is similar. If G is generated by $h_1, h_2, \cdots, h_r$ we define the subspace

$$X(g) = \{L(g), S_i(g), A_i(g), L(gh_i), i = 1, \cdots, r\}.$$

The topology on $X(g)$ is such that the L points are open, $\{L(g), S_1(g), \cdots, S_i(g)\}$ is the minimal open set of $S_i(g)$, and $\{L(g), S_1(g), \cdots, S_i(g), A_i(g), L(gh_i)\}$ is the minimal open set of $A_i(g)$. One can think of $X(g)$ as a linear body $S_1(g), S_2(g), \cdots, S_r(g)$ based at $L(g)$ with arms $A_i(g)$ attached along $L(g), S_1(g), \cdots, S_i(g)$ and open hands $L(gh_i)$. See Figure 3. X is obtained from $\bigcup \{X(g)\}, g \in G$, by identifying the open hand $L(gh_i)$ of $X(g)$

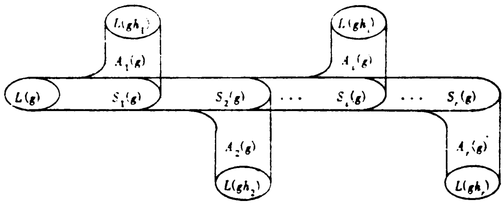

FIGURE 3. $X(g)$ for G with r generators.

with the base $L(gh_i)$ of $X(gh_i)$. The resulting X is clearly a T_0-space where every point has a finite minimal open set. If G has finite order n, X has $n(2r+1)$ points.

The proof that the group of homeomorphisms is isomorphic to G proceeds like the dihedral group. The function $\phi(g)$ which preserves the type and left multiplies the labels of points by g is a homeomorphism. Any homeomorphism must preserve the type of a point. Homeomorphisms are determined by the image of one point since any two L points may be joined by a chain of overlapping minimal open sets of the A_i. Therefore $g \rightarrow \phi(g)$ is again an isomorphism between G and the group of homeomorphisms of X.

4. **Covering spaces.** This construction can be viewed in terms of covering the spaces in the following way. Suppose G is given in terms of generators $h_1, \cdots, h_r$ and a set of relators $\{R_\alpha\}$. Define X_r to be the space obtained from $X(e)$ by adding a unit interval I_i from $L(h_i)$ to $L(e)$. Thus $L(h_i)=0 \in I_i$ and $L(e)=1 \in I_i$ for $i=1, \cdots, r$. Then basic open sets containing $A_i(e)$ are of the form $\{(a, 1] \subset I_i, S_1(e), \cdots, S_i(e), A_i(e), [0, b) \subset I_i\}$, etc. Theorem 1 of **[6]** shows that X_r has the same weak homotopy type as a wedge of r circles. Thus $\Pi_1(X_r, L(e))$ is the free group on r generators. Think of h_i as the element represented by a path from $L(e)$, through $S_1(e)$, $S_2(e), \cdots, S_i(e)$, $A_i(e)$, $L(h_i)$ then along I_i back to $L(e)$. Clearly the $h_1, \cdots, h_r$ are a free set of generators of $\Pi_1(X_r, L(e))$. Let H be the normal subgroup generated by the relators $\{R_\alpha\}$. The covering space X_H corresponding to the subgroup H is then of the same homotopy type as X. In fact X_H is just the union $\bigcup \{X(g), g \in G\}$ with a unit interval $I_{g,i}$ added for each g and i with $L(gh_i)$ in $X(g)$ being 0 in $I_{g,i}$ and $L(gh_i)$ in $X(gh_i)$ being 1 in $I_{g,i}$.

REFERENCES

1. G. Birkhoff, *On groups of automorphisms*, Rev. Un. Mat. Argentina **11** (1946), 155–157. (Spanish) MR **7**, 411.

2. R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compositio Math. **6** (1938), 239–250.

3. R. Frucht, *Graphs of degree three with a given abstract group*, Canad. J. Math. **1** (1949), 365–378. MR **11**, 377.

4. J. de Groot, *Groups represented by homeomorphism groups*. I, Math. Ann. **138** (1959), 80–102. MR **22** #9959.

5. F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass., 1969. MR **41** #1566.

6. M. C. McCord, *Singular homology groups and homotopy groups of finite topological spaces*, Duke Math. J. **33** (1966), 465–474. MR **33** #4930.

7. R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. **123** (1966), 325–340. MR **33** #3247.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA 68508