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ISOTOPIC CLOSED NONCONJUGATE BRAIDS[1]

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ABSTRACT. J. S. Birman has conjectured that, when a knot is represented by a closed braid on a minimal number n of strands, the conjugacy class of the braid exhausts the set of braids in B_n closing to define the knot. Counterexamples are given to disprove the conjecture, even when it is weakened to refer only to oriented knots.

1. **Introduction.** It is easy to see that a knot represented by the closure of a braid in the n-strand braid group B_n can also be represented by the closure of a braid in B_m for each $m>n$. Having made the above observation, J. S. Birman conjectured [1] that if A is a knot and n is the smallest integer such that there is a braid $\alpha\in B_n$ that closes to define the knot A, then the conjugacy class of α in B_n gives the totality of braids in B_n that close to define knots of the same isotopy type as α. Leading to this conjecture is a proof that turning over a braid leads to a conjugate braid. Let B_n be presented thus:

$$\langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1}; \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geqq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle;$$

then the turning-over operation $\triangle$ can be described thus

$$\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}\cdots\sigma_{i_k}^{\varepsilon_k} \xrightarrow{\triangle} \sigma_{n-i_1}^{\varepsilon_1}\sigma_{n-i_2}^{\varepsilon_2}\cdots\sigma_{n-i_k}^{\varepsilon_k}.$$

Since the conjecture is trivially true for $n=2$, we consider only the case $n\geqq 3$. We show by Counterexample 1 that a second operation, turning upside down, may not lead to a conjugate braid. We define the second operation thus:

$$\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}\cdots\sigma_{i_k}^{\varepsilon_k} \xrightarrow{\nabla} \sigma_{n-i_k}^{\varepsilon_k}\cdots\sigma_{n-i_2}^{\varepsilon_2}\sigma_{n-i_1}^{\varepsilon_1}.$$

It is obvious that if a braid α closes to define a knot A, then both the braids $\triangle\alpha$ and $\nabla\alpha$ close to define a knot of the isotopy type of A. The original conjecture must therefore be weakened to concern only oriented knots.

We show by Counterexample 2 that the weakened conjecture admits an infinite class of counterexamples within B_n for each $n\geqq 4$.

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2. **Counterexample 1.** The knot 6_3 in Reidemeister's table can be represented by the closure of braids in B_3 but not in B_2. We consider the braid $\gamma=\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2\in B_3$, which closes to define 6_3. And we show that $\nabla\gamma=\sigma_1\sigma_2^{-2}\sigma_1^2\sigma_2^{-1}\in B_3$ is not conjugate to γ. When γ and $\nabla\gamma$ are expressed within the presentation of B_3 as $\langle a, b; a^3=b^2\rangle$, where $a=\sigma_1\sigma_2$ and $b=\sigma_1\sigma_2\sigma_1$, they can be written thus:

$$\gamma = (baba^2ba^2bababa^2)a^{-18}, \qquad \nabla\gamma = (a^2bababa^2ba^2bab)a^{-18}.$$

Since the conjugacy classes in this normal form are just the cyclic permutations of the bracketed factors, the classes of γ and $\nabla\gamma$ are apparently distinct.

3. **Counterexample 2.** It will be observed that the construction of the counterexamples to the weakened conjecture depends on the oriented knots' being both composite and not representable by the closure of a braid of B_3. Within B_4, let $\alpha=\sigma_1^m\sigma_2^n\sigma_3^p$ and $\beta=\sigma_1^m\sigma_2^p\sigma_3^n$ with m, n, p all different, odd, and at least three in absolute value. Let A, B be the oriented knots defined by orienting from top to bottom and closing α and β respectively. Both A and B can be formed by the composition of oriented torus knots of types $(2, m)$, $(2, n)$, $(2, p)$. By the commutativity of the composition of oriented knots, A and B are isotopic: we show that neither can be represented as the closure of an element of B_2 or B_3 and secondly that α and β are not conjugate in B_4.

Suppose that A is represented in B_n. Then the group of A is generated by at most n Wirtinger generators with $n-1$ defining relations. Therefore the length of the chain of ideals[2] of A is at most $n-1$. But the length of the chain of ideals of a composite knot is the sum of the lengths of the chains of ideals of its components by **[2]**. And the length of the chain of ideals of each of the three components of A is one. Therefore $3\leqq n-1$ which implies that $n\geqq 4$. The same argument holds for B.

To show that $\sigma_1^m\sigma_2^n\sigma_3^p$ is not conjugate to $\sigma_1^m\sigma_2^p\sigma_3^n$ in B_4, let N be the normal closure of $\sigma_1\sigma_3^{-1}$ in B_4 and let $\phi: B_4\rightarrow B_4/N\cong B_3$ be the natural homomorphism. Then

$$\phi(\sigma_1^m\sigma_2^n\sigma_3^p) = \sigma_1^m\sigma_2^n\sigma_1^p \quad\text{and}\quad \phi(\sigma_1^m\sigma_2^p\sigma_3^n) = \sigma_1^m\sigma_2^p\sigma_1^n.$$

But $\sigma_1^{m+p}\sigma_2^n$ and $\sigma_1^{m+n}\sigma_2^p$ are not conjugate in B_3 since when they are closed they do not define isotopic links but rather the torus knots of type $(2, n)$ and $(2, p)$ respectively linked to an unknotted circle. Therefore α is not conjugate to β in B_4.

The existence of a similar infinite family of counterexamples in B_n for each larger n is obvious. These examples further stress the need to treat

[2] For the definition, see **[2**, p. 259**]**.

oriented rather than unoriented knots since α can be used in place of γ in §2.

REFERENCES

1. J. S. Birman, *Non-conjugate braids can define isotopic knots*, Comm. Pure Appl. Math. **22** (1969), 239–242. MR **39** #6298.

2. R. H. Fox, *On the total curvature of some tame knots*, Ann. of Math. (2) **52** (1950), 258–260. MR **12**, 273.

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