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HOMOTOPY TYPES OF THE DELETED PRODUCT OF UNIONS OF TWO SIMPLEXES

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ABSTRACT. If X is a space, let $X^*=X\times X-D$, where D is the diagonal. If f is a map on X to a space Y, let $X_f^*=\{(x,y)\in X^* \mid f(x)\neq f(y)\}$. In this paper we continue our investigation, begun in [6], of the homotopy types of X^* and X_f^*, and of a question due to Brahana [1, p. 236], as to when the homotopy types of X^* and X_f^* are the same. If X is the union of two nondisjoint simplexes, and if f is a simplicial map on X, we are able, using results and techniques developed in [6], to express the homotopy types of X^* and X_f^* in terms of spheres, and then to determine when the homotopy types of these spaces are the same.

1. **Introduction and notation.** If X is a finite polyhedron and f is a simplicial map on X, let

$$P(X^*) = \bigcup \{r \times s \mid r \text{ and } s \text{ are simplexes in } X \text{ with } r \cap s = \varnothing\},$$

and let $P(X_f^*)=\bigcup \{r\times s \mid r$ and s are simplexes in X with $f(r)\cap f(s)=\varnothing\}$. In [2, pp. 351–352], Hu has shown that $P(X^*)$ and X^* are homotopically equivalent, and in [3, p. 183] Patty has observed that $P(X_f^*)$ and X_f^* are homotopically equivalent.

The symbol $\langle v_0,\cdots,v_n\rangle$ will denote the n-simplex with vertices $v_0,\cdots,v_n$. We let $S^n=\{x\in E^{n+1} \mid |x|=1\}$. If X and Y are spaces, "$X\simeq Y$" will mean that X and Y are homotopically equivalent.

2. **The results.** For the remainder of this paper, we let $X=A\cup B$, where $A=\langle v_0,\cdots,v_k,v_{k+1},\cdots,v_n\rangle$ and $B=\langle v_0,\cdots,v_k,w_{k+1},\cdots,w_m\rangle$ are n- and m-simplexes, respectively, such that $A\cap B=\langle v_0,\cdots,v_k\rangle$ is a nonempty k-simplex, $k<n$, $k<m$.

THEOREM 1. *Let $f\colon A\to Y$ be a simplicial map, Y a polyhedron. Then $P(A_f^*)\simeq P(f(A)^*)\simeq S^{q-1}$, where $q=\dim f(A)$.*

PROOF. There is a face w of A, say $w=\langle v_0,\cdots,v_q\rangle$, such that $f(w)=f(A)$ and $f|w$ is a homeomorphism. Hence, $P(w_f^*)=P(w^*)\simeq P(f(w)^*)=P(f(A)^*)$.

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Let $g: A \to w$ be the simplicial map defined by $g(v) = w \cap f^{-1}f(v)$ for each vertex v in A. Then $g^2 = g$, and g takes A into A. Hence, $P(A_g^*) \simeq P(g(A)^*) = P(w^*)$ (Corollary 2.1 of [6]).

It is straightforward to show that $P(A_g^*) = P(A_f^*)$, so we have $P(A_f^*) \simeq P(f(A)^*)$. Since $f(A)$ is a simplex of dimension q, the remainder of the theorem follows from Corollary 1 of [4].

Let f be a simplicial map on X to a polyhedron Y. Let

$$D_1 = \bigcup \{r \times s \mid r \text{ is a face of } A, s \text{ is a face of } B, \text{ and } f(r) \cap f(s) = \varnothing\},$$

and let $D_2 = \bigcup \{r \times s \mid r \text{ is a face of } B, s \text{ is a face of } A, \text{ and } f(r) \cap f(s) = \varnothing\}$. Note that $P(X_f^*) = P(A_f^*) \cup P(B_f^*) \cup D_1 \cup D_2$.

LEMMA 1. *If $f(B)$ is not contained in $f(A)$, then D_1, D_2, $D_1 \cap P(B_f^*)$ and $D_2 \cap P(B_f^*)$, are all contractible.*

PROOF. Let r and s be simplexes in X with $r \times s \subset D_1$. There exists a vertex $v_B \in B$ such that $f(v_B) \notin f(A)$. We may assume $v_B \in s$. Then if g is the linear map defined on D_1 by $g(r \times s) = r \times \{v_B\}$, it is clear that $D_1 \simeq g(D_1)$. Now if $r \times \{v_B\}$ is a cell in $g(D_1)$, we may assume $v_0 \in r$. Then it is clear that $g(D_1) \simeq \{(v_0, v_B)\}$. Therefore, D_1 is contractible. Since D_1 and D_2 are homeomorphic, D_2 is also contractible.

Now observe that if $r \times s$ is a cell in $D_1 \cap P(B_f^*)$, r is a face of $A \cap B$, and s is a face of B, and since $v_0 \in A \cap B$, we may contract $D_1 \cap P(B_f^*)$ to (v_0, v_B) by the same argument as above. Similarly, $D_2 \cap P(B_f^*)$ is contractible.

The proof of the following lemma is straightforward and hence is left to the reader.

LEMMA 2. *If $f(A) \neq f(A \cap B)$, then $D_1 \cap P(A_f^*)$ and $D_2 \cap P(A_f^*)$ are contractible.*

LEMMA 3. *If $f(B)$ is not a subset of $f(A)$, then $P(B_f^*) \simeq P(B_f^*) \cup D_1 \cup D_2$ If, in addition, $f(A) \neq f(A \cap B)$, then $P(A_f^*) \simeq P(A_f^*) \cup D_1 \cup D_2$.*

PROOF. By Lemma 1, $P(B_f^*) \simeq P(B_f^*) \cup D_1$. Clearly, $D_1 \cap D_2 \subset P(B_f^*) \cap D_2$, so $(P(B_f^*) \cup D_1) \cap D_2 = P(B_f^*) \cap D_2$, which is contractible by Lemma 1. Then since D_2 is contractible, $P(B_f^*) \simeq P(B_f^*) \cup D_1 \simeq P(B_f^*) \cup D_1 \cup D_2$. Similarly, using Lemma 2, we have $P(A_f^*) \simeq P(A_f^*) \cup D_1 \cup D_2$.

LEMMA 4. *If there is a vertex $v_A \in A$ such that $f(v_A) \notin f(A \cap B)$, then $(D_1 \cup D_2) \cap P(A_f^*) \simeq P(\langle v_0, \cdots, v_k, v_A \rangle_f^*)$.*

PROOF. The proof is the same as the proof of Lemma 4 of [6].

THEOREM 2. *Let f be a simplicial map on X. Suppose there exist vertices $v_A \in A$ and $v_B \in B$ such that $f(v_A) \notin f(A \cap B)$ and $f(v_B) \notin f(A)$. If $n = k+1$, then $P(X_f^*) \simeq S^r$, where $r+1 = \dim f(B)$; if $m = k+1$, then $P(X_f^*) \simeq S^q$, where*

$q+1=\dim f(A)$; *if* $m, n>k+1$, *then* $P(X_f^*)\simeq S^q\cup S^r$, *where* $S^q\cap S^r\simeq S^t$, $t+1=\dim f(\langle v_0, \cdots, v_k, v_A\rangle)$.

PROOF. If $n=k+1$, then $P(A_f^*)\subset P(B_f^*)\cup D_1\cup D_2$. Then $P(X_f^*)=P(B_f^*)\cup D_1\cup D_2$. But $P(B_f^*)\cup D_1\cup D_2\simeq P(B_f^*)\simeq S^r$ from Theorem 1 and Lemma 3. Hence, $P(X_f^*)\simeq S^r$. Similarly, if $m=k+1$, $P(X_f^*)\simeq S^q$.

Now assume $m, n>k+1$. By Theorem 1, $P(A_f^*)\simeq S^q$ and $P(B_f^*)\simeq S^r$. Then by Lemma 3, $P(B_f^*)\cup D_1\cup D_2\simeq S^r$. Then

$$\begin{aligned}(P(B_f^*) \cup D_1 \cup D_2) \cap P(A_f^*)&\\ &= [P(A_f^*) \cap P(B_f^*)] \cup [(D_1 \cup D_2) \cap P(A_f^*)]\\ &= [P((A \cap B)_f^*)] \cup [(D_1 \cup D_2) \cap P(A_f^*)]\\ &= (D_1 \cup D_2) \cap P(A_f^*)\\ &\simeq P(\langle v_0, \cdots, v_k, v_A\rangle_f^*) \quad \text{by Lemma 4}\\ &\simeq S^t \quad\quad \text{by Theorem 1.}\end{aligned}$$

This proves the theorem.

COROLLARY 2.1. *If* $n=k=1$, $P(X^*)\simeq S^{m-1}$; *if* $m=k+1$, $P(X^*)\simeq S^{n-1}$; *if* $m, n>k+1$, $P(X^*)\simeq S^{m-1}\cup S^{n-1}$, *with* $S^{m-1}\cap S^{n-1}\simeq S^k$.

PROOF. Let f be the identity map in Theorem 2.

We should note that the results of Corollary 2.1 for the cases $m=k+1$ or $n=k+1$ are special cases of Theorem 6 of [**5**].

Note further that Corollary 2.1 allows us to compute the homology groups of $P(X^*)$ easily.

We now need to consider the cases where both of the conditions (1) $f(A)\neq f(A\cap B)$ and $f(B)\nsubseteq f(A)$ and (2) $f(B)\neq f(A\cap B)$ and $f(A)\nsubseteq f(B)$ fail simultaneously. There are three such possibilities: (i) $f(A)=f(A\cap B)$, (ii) $f(B)=f(A\cap B)$, (iii) $f(A)=f(B)\neq f(A\cap B)$.

THEOREM 3. *If* $f(A)=f(A\cap B)$, *then* $P(X_f^*)\simeq P(B_f^*)$; *if* $f(B)=f(A\cap B)$, *then* $P(X_f^*)\simeq P(A_f^*)$.

PROOF. Suppose $f(A)=f(A\cap B)$. There is a face u of B such that $f(u)=f(B)=f(X)$, $f(u\cap A)=f(A\cap B)=f(A)$, and $f|u$ is a homeomorphism. Let $g:X\to X$ be the simplicial map defined by $g(v)=u\cap f^{-1}f(v)$. Then by Corollary 2.1 of [**6**], $P(X_g^*)\simeq P(g(X)^*)=P(g(B)^*)\simeq P(B_g^*)$. It is straightforward to show that $P(X_g^*)=P(X_f^*)$ and $P(B_g^*)=P(B_f^*)$ since $f=fg$, and $f|u$ is one-to-one. Hence, $P(X_f^*)\simeq P(B_f^*)$.

Similarly, if $f(B)=f(A\cap B)$, then $P(X_f^*)\simeq P(A_f^*)$.

COROLLARY 3.1. *If* $f(A)=f(A\cap B)$, *then* $P(X_f^*)\simeq S^r$, *where* $r+1=\dim f(B)$; *if* $f(B)=f(A\cap B)$, *then* $P(X_f^*)\simeq S^q$, *where* $q+1=\dim f(A)$.

PROOF. This is a direct consequence of Theorem 1 and Theorem 3.

Now suppose $f(A)=f(B)\neq f(A\cap B)$. Then we may choose faces $r_1=\langle u_0,\cdots,u_p,u_{p+1},\cdots,u_q\rangle\subset A$ and $r_2=\langle u_0,\cdots,u_p,t_{p+1},\cdots,t_q\rangle\subset B$ such that $f(r_1)=f(A)$, $f(r_2)=f(B)$, $r_1\cap r_2=\langle u_0,\cdots,u_p\rangle$ with $f(r_1\cap r_2)=f(A\cap B)$, and $f|r_i$ is a homeomorphism, $i=1,2$.

LEMMA 5. *Let $W=r_1\cup r_2$. Then $P(W_f^*)\simeq P(X_f^*)$.*

PROOF. The proof is the same as the proof of Lemma 7 of **[6]**.

We may now assume $f(W)=r_1$ since $f(W)=f(r_1)$ and $f|r_1$ is a homeomorphism.

LEMMA 6. *If $p=0$ and $q=p+1$, $P(W_f^*)$ has the homotopy type of four points. If $p\geqq 1$ and $q=p+1$, the $(q-1)$st homology group of $P(W_f^*)$ is isomorphic to the direct sum of three copies of the integers. If $q>p+1$, the $(q-1)$st homology group of $P(W_f^*)$ is isomorphic to a group which contains the direct sum of three copies of the integers as a proper subgroup.*

PROOF. If $p=0$ and $q=p+1$, the result is trivial. Otherwise, since $f|r_i$ is a homeomorphism, $i=1,2$, and $f(W)=f(r_1)=r_1$, the result follows directly from Case I of the proof of Lemma 9 of **[6]** since $P(r_{1f}^*)\simeq P(r_{2f}^*)\simeq S^{q-1}$ by Theorem 1.

THEOREM 4. *If $f(A)=f(B)\neq f(A\cap B)$, then $P(X^*)\not\simeq P(X_f^*)$.*

PROOF. If $m=k+1$, $P(X^*)\simeq S^{n-1}$ by Corollary 3.1. But by Lemma 6, $P(X_f^*)\not\simeq S^{n-1}$. Similarly, if $n=k+1$, $P(X^*)\not\simeq P(X_f^*)$.

Suppose m, $n>k+1$. Then by Lemma 6, some nonzero homology group of $P(X_f^*)$ contains a direct summand of three copies of the integers as a subgroup. By Corollary 2.1, some nonzero homology group of $P(X^*)$ contains a subgroup isomorphic to a direct summand of three copies of the integers if and only if $m=n=k+2$, and in this case, the $(m-1)$st homology group of $P(X^*)$ is isomorphic to the direct sum of three copies of the integers. But by Lemma 6, the $(m-1)$st homology group of $P(X_f^*)$ is isomorphic to the direct sum of three copies of the integers if and only if $\dim f(X)=m=1+\dim f(A\cap B)$, which is impossible if $m=k+2$.

COROLLARY 4.1. *Suppose f is any simplicial map on X. Then $P(X^*)\simeq P(X_f^*)$ if and only if* (1) $f(A)=f(A\cap B)$, $\dim f(X)=m$, *and* $n=k+1$; (2) $f(B)=f(A\cap B)$, $\dim f(X)=n$, *and* $m=k+1$; (3) $f(B)\nsubseteq f(A)$, $f(A)\neq f(A\cap B)$, $n=k+1$, *and* $\dim f(B)=m$; (4) $f(A)\nsubseteq f(B)$, $f(B)=f(A\cap B)$, $m=k+1$, *and* $\dim f(A)=n$; (5) $f(B)\nsubseteq f(A)$, $f(A)=f(A\cap B)$, m, $n>k+1$, $\dim f(B)=m$, $\dim f(A)=n$, *or* $\dim f(B)=n$, $\dim f(A)=m$; *or* (6) $f(A)\nsubseteq f(B)$, $f(B)\neq f(A\cap B)$, $m,n>k+1$, $\dim f(B)=m$, $\dim f(A)=n$, *or* $\dim f(B)=n$, $\dim f(A)=m$.

PROOF. This follows directly from the preceding results.

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