HOMOTOPY TYPES OF THE DELETED PRODUCT OF UNIONS OF TWO SIMPLEXES

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Abstract. If \( X \) is a space, let \( X^*=X\times X-D \), where \( D \) is the diagonal. If \( f \) is a map on \( X \) to a space \( Y \), let \( X_f^*=\{(x,y)\in X^*|f(x)\neq f(y)\} \). In this paper we continue our investigation, begun in [6], of the homotopy types of \( X^* \) and \( X_f^* \), and of a question due to Brahana [1, p. 236], as to when the homotopy types of \( X^* \) and \( X_f^* \) are the same. If \( X \) is the union of two nondisjoint simplices, and if \( f \) is a simplicial map on \( X \), we are able, using results and techniques developed in [6], to express the homotopy types of \( X^* \) and \( X_f^* \) in terms of spheres, and then to determine when the homotopy types of these spaces are the same.

1. Introduction and notation. If \( X \) is a finite polyhedron and \( f \) is a simplicial map on \( X \), let

\[
P(X^*) = \bigcup \{r \times s | r \text{ and } s \text{ are simplexes in } X \text{ with } r \cap s = \emptyset \},
\]

and let \( P(X_f^*) = \bigcup \{r \times s | r \text{ and } s \text{ are simplexes in } X \text{ with } f(r) \cap f(s) = \emptyset \}. \)

In [2, pp. 351–352], Hu has shown that \( P(X^*) \) and \( X^* \) are homotopically equivalent, and in [3, p. 183] Patty has observed that \( P(X_f^*) \) and \( X_f^* \) are homotopically equivalent.

The symbol \((v_0, \ldots, v_n)\) will denote the \( n \)-simplex with vertices \( v_0, \ldots, v_n \). We let \( S^n = \{x \in \mathbb{E}^{n+1} | |x| = 1 \} \). If \( X \) and \( Y \) are spaces, "\( X \approx Y \)" will mean that \( X \) and \( Y \) are homotopically equivalent.

2. The results. For the remainder of this paper, we let \( X=A \cup B \), where \( A=(v_0, v_1, \ldots, v_n) \) and \( B=(v_0, \ldots, v_m, v_{k+1}, \ldots, v_m) \) are \( n \)- and \( m \)-simplexes, respectively, such that \( A \cap B=(v_0, \ldots, v_k) \) is a non-empty \( k \)-simplex, \( k<n, k<m \).

Theorem 1. Let \( f:A \to Y \) be a simplicial map, \( Y \) a polyhedron. Then \( P(A_f^*) \approx P(f(A)^*) \approx S^{q-1} \), where \( q=\dim f(A) \).

Proof. There is a face \( w \) of \( A \), say \( w=(v_0, \ldots, v_w) \), such that \( f(w) \) = \( f(A) \) and \( f|w \) is a homeomorphism. Hence, \( P(w_f^*) \approx P(w^*) \approx P(f(w)^*) = P(f(A)^*) \).

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Let \( g : A \rightarrow w \) be the simplicial map defined by \( g(v) = w \cap f^{-1}(v) \) for each vertex \( v \) in \( A \). Then \( g^* = g \), and \( g \) takes \( A \) into \( A \). Hence, \( P(A^*_A) \simeq P(g(A)^*_A) = P(w^*) \) (Corollary 2.1 of [6]).

It is straightforward to show that \( P(A^*_A) = P(A^*_A) \), so we have \( P(A^*_A) \simeq P(f(A)^*_A) \). Since \( f(A) \) is a simplex of dimension \( q \), the remainder of the theorem follows from Corollary 1 of [4].

Let \( f \) be a simplicial map on \( X \) to a polyhedron \( Y \). Let

\[
\begin{align*}
D_1 &= \bigcup \{ r \times s \mid r \text{ is a face of } A, s \text{ is a face of } B, \text{ and } \cap f(r) = \emptyset \}, \\
D_2 &= \bigcup \{ r \times s \mid r \text{ is a face of } B, s \text{ is a face of } A, \text{ and } \cap f(s) = \emptyset \}.
\end{align*}
\]

Note that \( P(A^*_A) = P(A^*_A) \cup P(B^*_B) \cup D_1 \cup D_2 \).

**Lemma 1.** If \( f(B) \) is not contained in \( f(A) \), then \( D_1, D_2, D_1 \cap P(B^*_B) \), and \( D_2 \cap P(B^*_B) \), are all contractible.

**Proof.** Let \( r \) and \( s \) be simplexes in \( X \) with \( r \cap s \subseteq D_1 \). There exists a vertex \( v_B \in B \) such that \( f(v_B) \not\subseteq f(A) \). We may assume \( v_B \in f(A) \). Then if \( g \) is the linear map defined on \( D_1 \) by \( g(r \times s) = r \times \{ v_B \} \), it is clear that \( D_1 \simeq g(D_1) \). Now if \( r \times \{ v_B \} \) is a cell in \( g(D_1) \), we may assume \( v_B \in r \). Then it is clear that \( g(D_1) \simeq \{ (r_0, v_B) \} \). Therefore, \( D_1 \) is contractible. Since \( D_1 \) and \( D_2 \) are homeomorphic, \( D_2 \) is also contractible.

Now observe that if \( r \cap s \) is a cell in \( D_1 \cap P(B^*_B) \), \( r \) is a face of \( A \) \( \cap B \), and \( s \) is a face of \( B \), and since \( v_B \in A \cap B \), we may contract \( D_1 \cap P(B^*_B) \) to \( (v_B, v_B) \) by the same argument as above. Similarly, \( D_2 \cap P(B^*_B) \) is contractible.

The proof of the following lemma is straightforward and hence is left to the reader.

**Lemma 2.** If \( f(A) \neq f(A \cap B) \), then \( D_1 \cap P(A^*_A) \) and \( D_2 \cap P(A^*_A) \) are contractible.

**Lemma 3.** If \( f(B) \) is not a subset of \( f(A) \), then \( P(B^*_B) \simeq P(B^*_B) \cup D_1 \cup D_2 \) as if, in addition, \( f(A) \neq f(A \cap B) \), then \( P(A^*_A) \simeq P(A^*_A) \cup D_1 \cup D_2 \).

**Proof.** By Lemma 1, \( P(B^*_B) \simeq P(B^*_B) \cup D_1 \). Clearly, \( D_1 \cup D_2 \subseteq P(B^*_B) \cap D_2 \), so \( (P(B^*_B) \cup D_1) \cap D_2 = P(B^*_B) \cap D_2 \), which is contractible by Lemma 1. Then since \( D_2 \) is contractible, \( P(B^*_B) \simeq P(B^*_B) \cup D_1 \simeq P(B^*_B) \cup D_1 \cup D_2 \) Similarly, using Lemma 2, we have \( P(A^*_A) \simeq P(A^*_A) \cup D_1 \cup D_2 \).

**Lemma 4.** If there is a vertex \( v_A \in A \) such that \( f(v_A) \not\subseteq f(A \cap B) \), then \( (D_1 \cup D_2) \cap P(A^*_A) \simeq P(v_0, \ldots, v_k, v_A^*_A) \).

**Proof.** The proof is the same as the proof of Lemma 4 of [6].

**Theorem 2.** Let \( f \) be a simplicial map on \( X \). Suppose there exist vertices \( v_A \in A \) and \( v_B \in B \) such that \( f(v_A) \not\subseteq f(A \cap B) \) and \( f(v_B) \not\subseteq f(A) \). If \( n = k + 1 \), then \( P(X^*_A) \simeq S^r \), where \( r + 1 = \dim f(B) \); if \( m = k + 1 \), then \( P(X^*_A) \simeq S^q \), where
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\[ q + 1 = \dim f(A); \text{ if } m, n > k+1, \text{ then } P(X^*_r) \simeq S^q \cup S^r, \text{ where } S^q \cap S^r \simeq S^t, \]

\[ t + 1 = \dim f((v_0, \cdots, v_k, v_{k+1})). \]

**Proof.** If \( n = k + 1 \), then \( P(A^*_r) \subset P(B^*_r) \cup D_1 \cup D_2 \). Then \( P(X^*_r) = P(B^*_r) \cup D_1 \cup D_2 \). But \( P(B^*_r) \cup D_1 \cup D_2 \simeq P(B^*_r) \simeq S^r \) from Theorem 1 and Lemma 3. Hence, \( P(X^*_r) \simeq S^r \). Similarly, if \( m = k + 1 \), \( P(X^*_r) \simeq S^q \).

Now assume \( m, n > k + 1 \). By Theorem 1, \( P(A^*_r) \simeq S^q \) and \( P(B^*_r) \simeq S^r \). Then by Lemma 3, \( P(B^*_r) \cup D_1 \cup D_2 \simeq S^r \). Then

\[
(P(B^*_r) \cup D_1 \cup D_2) \cap P(A^*_r)
\]

\[ = [P(A^*_r) \cap P(B^*_r)] \cup [(D_1 \cup D_2) \cap P(A^*_r)]
\]

\[ = [P((A \cap B)^*_r)] \cup [(D_1 \cup D_2) \cap P(A^*_r)]
\]

\[ = (D_1 \cup D_2) \cap P(A^*_r)
\]

\[ \simeq P((v_0, \cdots, v_k, v_{k+1})) \text{ by Lemma 4}
\]

\[ \simeq S^r \text{ by Theorem 1.} \]

This proves the theorem.

**Corollary 2.1.** If \( n = k + 1 \), \( P(X^*_r) \simeq S^{m-1} \); if \( m = k + 1 \), \( P(X^*_r) \simeq S^{n-1} \); if \( m, n > k + 1 \), \( P(X^*_r) \simeq S^{m-1} \cup S^{n-1}, \text{ with } S^{m-1} \cap S^{n-1} \simeq S^q \).

**Proof.** Let \( f \) be the identity map in Theorem 2.

We should note that the results of Corollary 2.1 for the cases \( m = k + 1 \) or \( n = k + 1 \) are special cases of Theorem 6 of [5].

Note further that Corollary 2.1 allows us to compute the homology groups of \( P(X^*_r) \) easily.

We now need to consider the cases where both of the conditions (1) \( f(A) \neq f(A \cap B) \) and \( f(B) \neq f(A) \) and (2) \( f(B) \neq f(A \cap B) \) and \( f(A) \neq f(B) \) fail simultaneously. There are three such possibilities: (i) \( f(A) = f(A \cap B) \), (ii) \( f(B) = f(A \cap B) \), (iii) \( f(A) = f(B) \neq f(A \cap B) \).

**Theorem 3.** If \( f(A) = f(A \cap B) \), then \( P(X^*_r) \simeq P(B^*_r) \); if \( f(B) = f(A \cap B) \), then \( P(X^*_r) \simeq P(A^*_r) \).

**Proof.** Suppose \( f(A) = f(A \cap B) \). There is a face \( u \) of \( B \) such that \( f(u) = f(B) = f(X) \), \( f(u \cap A) = f(A \cap B) = f(A) \), and \( f|u \) is a homeomorphism. Let \( g: X \to X \) be the simplicial map defined by \( g(v) = u \cap f^{-1}f(v) \). Then by Corollary 2.1 of [6], \( P(X^*_r) \simeq P(g(X^*_r)) \simeq P(B^*_r) \). It is straightforward to show that \( P(X^*_r) \simeq P(B^*_r) \) and \( P(B^*_r) \simeq P(A^*_r) \) since \( f = fg \), and \( f|u \) is one-to-one. Hence, \( P(X^*_r) \simeq P(B^*_r) \).

Similarly, if \( f(B) = f(A \cap B) \), then \( P(X^*_r) \simeq P(A^*_r) \).

**Corollary 3.1.** If \( f(A) = f(A \cap B) \), then \( P(X^*_r) \simeq S^r \), where \( r + 1 = \dim f(B) \); if \( f(B) = f(A \cap B) \), then \( P(X^*_r) \simeq S^q \), where \( q + 1 = \dim f(A) \).

**Proof.** This is a direct consequence of Theorem 1 and Theorem 3.
Now suppose \( f(A) = f(B) \neq f(A \cap B) \). Then we may choose faces 
\[ r_1 = (u_0, \ldots, u_p, \ldots, u_p) \subseteq A \] and 
\[ r_2 = (u_0, \ldots, u_p, v_{p+1}, \ldots, v_q) \subseteq B \] 
such that \( f(r_1) = f(A), f(r_2) = f(B), r_1 \cap r_2 = (u_0, \ldots, u_p) \) with 
\( f(r_1 \cap r_2) = f(A \cap B) \), and \( f|_{r_i} \) is a homeomorphism, \( i = 1, 2 \).

**Lemma 5.** Let \( W = r_1 \cup r_2 \). Then \( P(W) \simeq P(X_f) \).

**Proof.** The proof is the same as the proof of Lemma 7 of [6].

We may now assume \( f(W) = r_1 \) since \( f(W) = f(r_1) \) and \( f|_{r_1} \) is a homeomorphism.

**Lemma 6.** If \( p = 0 \) and \( q = p + 1 \), \( P(W^p) \) has the homotopy type of four points. If \( p \geq 1 \) and \( q = p + 1 \), the \((q-1)st\) homology group of \( P(W^p) \) is isomorphic to the direct sum of three copies of the integers. If \( q > p + 1 \), the \((q-1)st\) homology group of \( P(W^p) \) is isomorphic to a group which contains the direct sum of three copies of the integers as a proper subgroup.

**Proof.** If \( p = 0 \) and \( q = p + 1 \), the result is trivial. Otherwise, since \( f|_{r_1} \) is a homeomorphism, \( i = 1, 2 \), and \( f(W) = f(r_1) = r_1 \), the result follows directly from Case I of the proof of Lemma 9 of [6] since \( P(r^p_1) \simeq P(r^p_2) \simeq S^{n-1} \) by Theorem 1.

**Theorem 4.** If \( f(A) = f(B) \neq f(A \cap B) \), then \( P(X^*) \cong P(X_f) \).

**Proof.** If \( m = k + 1 \), \( P(X^*) \cong S^{n-1} \) by Corollary 3.1. But by Lemma 6, \( P(X_f^p) \cong S^{n-1} \). Similarly, if \( n = k + 1 \), \( P(X^*) \cong P(X_f^p) \).

Suppose \( m, n > k + 1 \). Then by Lemma 6, some nonzero homology group of \( P(X_f^p) \) contains a direct summand of three copies of the integers as a subgroup. By Corollary 2.1, some nonzero homology group of \( P(X^*) \) contains a subgroup isomorphic to a direct summand of three copies of the integers if and only if \( m = n = k + 2 \), and in this case, the \((m-1)st\) homology group of \( P(X^*) \) is isomorphic to the direct sum of three copies of the integers. But by Lemma 6, the \((m-1)st\) homology group of \( P(X_f^p) \) is isomorphic to the direct sum of three copies of the integers if and only if \( \dim f(X) = m = k + 2 \).

**Corollary 4.1.** Suppose \( f \) is any simplicial map on \( X \). Then \( P(X^*) \cong P(X_f^p) \) if and only if

1. \( f(A) = f(A \cap B) \), \( \dim f(X) = m \), and \( n = k + 1 \);
2. \( f(B) = f(A \cap B) \), \( \dim f(X) = n \), and \( m = k + 1 \);
3. \( f(B) \neq f(A) \), \( f(A) \neq f(A \cap B) \), \( \dim f(B) = m \), and \( \dim f(A) = n \);
4. \( f(B) \neq f(A) \), \( f(A) = f(A \cap B) \), \( \dim f(B) = m \), and \( \dim f(A) = n \);
5. \( f(A) = f(B) \neq f(A \cap B) \), \( \dim f(B) = m \), and \( \dim f(A) = n \);
6. \( f(A) = f(B) \neq f(A \cap B) \), \( \dim f(B) = m \), and \( \dim f(A) = n \).

**Proof.** This follows directly from the preceding results.
BIBLIOGRAPHY


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