A CHARACTERIZATION OF GROUPS WITH ISOMORPHIC SUBGROUPS

DONALD W. MILLER

Abstract. The concept of normalizer is generalized to derive a characterization of groups $G$ which contain a proper subgroup isomorphic to $G$.

The problem of what groups have proper isomorphic subgroups has attracted the attention of a number of authors. Baer [1] obtained a characterization of groups without proper isomorphic subgroups, while Beaulmont [2], Kaplansky [4] and Clay [3] identified a number of classes of groups with proper isomorphic subgroups as well as several classes of groups without such subgroups. It is the purpose of this note to give an explicit characterization of groups with proper isomorphic subgroups.¹

Let $H$ be a subgroup of the group $G$, with normalizer $N = N_G(H)$, where $N = \{ x \in G | xH = Hx \}$. Define the left normalizer, $L = L_G(H)$, of $H$ in $G$ by $L = \{ x \in G | xH \subseteq Hx \}$ and the right normalizer, $R = R_G(H)$, of $H$ in $G$ by $R = \{ x \in G | Hx \subseteq xH \}$. The following observations are immediate consequences of these definitions.

Lemma. Let $H$ be a subgroup of a group $G$. Then:
(i) $L \cap R = N$;
(ii) $R = L^{-1} = \{ x^{-1} | x \in L \}$;
(iii) either $L = R = N$, or $L$, $R$ and $N$ are pairwise distinct.

The criterion referred to in the title can now be established.

Theorem. A group $H$ has an isomorphic proper subgroup if and only if $H$ is imbeddable in a group $G$ with the property that the left and right normalizers of $H$ in $G$ are distinct.

Proof. Suppose that $H$ is imbeddable in a group $G$ such that $L = L_G(H) \neq R = R_G(H)$. Then $L \supset N = N_G(H)$ so, selecting $a$ in $L \setminus N$ (the complement of $N$ in $L$), we have $aH \subset Ha$. Thus, $aHa^{-1}$ is a proper subgroup of $H$ which is isomorphic to $H$.

¹ The generalizations of the normalizer of a subgroup on which this characterization is based were suggested to the writer by Thomas Shores.
Conversely, let $H_1$ be a group which contains a proper isomorphic subgroup $H_0$, with $\varphi_0$ an isomorphism of $H_0$ onto $H_1$. Then

$$H_0 \varphi_0 = H_1 \supset H_0$$

so there exists a proper subgroup $H_{-1} = H_0 \varphi_0^{-1}$ of $H_0$ such that $H_{-1} \varphi_0 = H_0$. Consequently $\varphi_{-1} = \varphi_0 | H_{-1}$ (the restriction of $\varphi_0$ to $H_{-1}$) is an isomorphism of $H_{-1}$ onto $H_0$. Iteration of this process leads to a strictly decreasing chain of subgroups $H_1 \supset H_0 \supset H_{-1} \supset \cdots$ of $H_1$, together with a sequence of surjective isomorphisms $\varphi_i : H_i \rightarrow H_{i+1}$, $i = 0, -1, -2, \cdots$, such that

$$\varphi_i \big|_{H_{i-1}} = \varphi_{i-1}, \quad i = 0, -1, -2, \cdots$$

Let $A_1$ be a set disjoint from $H_1$ such that $|A_1| = |H_1| \setminus H_0|$, and let $\beta$ be a bijection from $H_1 \setminus H_0$ to $A_1$. Define a bijection $\varphi_1$ from $H_1$ to $H_2 = H_1 \cup A_1$ by

$$\varphi_1 |_{H_0} = \varphi_0, \quad \varphi_1 | (H_1 \setminus H_0) = \beta.$$ 

Extend the definition of multiplication from $H_1$ to $H_2$ by stipulating that

$$xy = ((x \varphi_1^{-1})(y \varphi_1^{-1})) \varphi_1, \quad \text{all } x, y \in H_2,$$

(which is clearly consistent with the existing multiplication in $H_1$). Then for all $x, y \in H_1$,

$$(x \varphi_1)(y \varphi_1) = ((x \varphi_1 \varphi_1^{-1})(y \varphi_1 \varphi_1^{-1})) \varphi_1 = (xy) \varphi_1,$$

so $\varphi_1$ is an isomorphism of $H_1$ onto $H_2$.

Repetition of this procedure yields a strictly increasing chain of groups $H_1 \subset H_2 \subset H_3 \subset \cdots$, together with a sequence of isomorphisms $\varphi_i : H_i \rightarrow H_{i+1}$, $i = 1, 2, 3, \cdots$, such that

$$\varphi_i \big|_{H_{i-1}} = \varphi_{i-1}, \quad i = 1, 2, 3, \cdots.$$ 

Now let $K = \bigcup_{i=0}^{\infty} H_i$, and define the transformation $\varphi$ of $K$ by $\varphi : x \mapsto \varphi_i x$ for all $x \in H_i$; by (1) and (2), $\varphi$ is well defined. It follows that $\varphi$ is an automorphism of $K$. Thus, $\varphi$ is induced by an inner automorphism, say $T(x)$, of the holomorph $G$ of $K$. Consequently,

$$\varphi^{-1} H_1 \varphi = H_1 T(x) = H_1, \quad \varphi H_2 \varphi = H_2 \supset H_1,$$

so $H_1 \varphi \supset \alpha H_1$. Therefore $\alpha \in L_G(H_1) \setminus N_G(H_1)$, so, by the lemma, $L_G(H_1) \neq R_G(H_1)$.

* Bibliography *


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA 68508