PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

# ALMOST CONTINUOUS REAL FUNCTIONS

K. R. KELLUM AND B. D. GARRETT

ABSTRACT. A blocking set of a function $f$ is a closed set which does not intersect $f$ but which intersects each continuous function with domain the same as $f$. It is shown that for each function which is not almost continuous, there exists a minimal blocking set. Using this property it is shown that there exists an almost continuous function with domain [0, 1] which is a $G_\delta$ set but is not of Baire Class 1, and that there exists an almost continuous function dense in the unit square.

Unless otherwise stated, all functions considered are real functions with domain a closed and bounded subset of the real line, $R$. No distinction is made between a function and its graph. If the function $f$ is a connected subset of the plane, $f$ is simply said to be a *connected function*. If each open set containing $f$ also contains a continuous function with the same domain as $f$, then the function $f$ is said to be *almost continuous*. Stallings [7] showed that if the function $f$ is almost continuous and has connected domain, then $f$ is connected. He stated as an open question the following. Is each connected function almost continuous? This question was answered in the negative by several authors ([1], [2], [4], [6]).

In [4], Jones and Thomas showed that there exists a function which is not of Baire Class 1, is a $G_\delta$ set, is connected but is not almost continuous. In Example 1 of the present paper, it is shown that there exists such a function which is almost continuous. Example 2 shows that there exists an almost continuous function which is a dense subset of the unit square. In [1], Brown gave an example of a connected but not almost continuous function dense in $[0, 1]\times R$.

Suppose $M$ is a subset of the plane. The $X$-projection of $M$ is denoted by $(M)_X$. The $Y$-projection of $M$ is denoted by $(M)_Y$. If $K\subset(M)_X$, $M_K$ denotes the part of $M$ with $X$-projection $K$.

The function $f$ is said to be of *Baire Class 1* if $f$ is the pointwise limit of a sequence of continuous functions.

Suppose that $D$ is a set containing the function $f$. The statement that the subset $C$ of $D$ is a *blocking set of $f$ relative to $D$* means that $C$ is closed

Received by the editors June 15, 1971.

*AMS 1970 subject classifications*. Primary 54C10, 54C30, 26A15; Secondary 26A21.

*Key words and phrases*. Connected function, connectivity function, almost continuous.

relative to $D$, $C$ contains no point of $f$, and $C$ intersects $g$ whenever $g$ is a continuous function such that $(g)_X=(f)_X$ and $g\subset D$. If no proper subset of $C$ is a blocking set of $f$ relative to $D$, $C$ is said to be a *minimal blocking set of $f$ relative to $D$.*

THEOREM 1. *Suppose the function $f$ has closed and bounded domain $H$, range $K$, and is not almost continuous. Then there exists a minimal blocking set $C$ of $f$ relative to $H\times K$. Further, $(C)_X$ is contained in a component of $H$ and, if $K$ is connected,* $\mathrm{Cl}((C)_X)$ *is connected.*

PROOF. Since $f$ is not almost continuous, there exists an open set $D$ containing $f$ such that $D$ contains no continuous function with domain $H$. Then $(H\times K)-(D\cap(H\times K))$ is a blocking set of $f$ relative to $H\times K$. Now, suppose $U$ is a monotonic collection of blocking sets of $f$ relative to $H\times K$ and $g$ is a continuous function with $(g)_X=H$ and $g\subset H\times K$. Then $g$ intersects each member of $U$. So, the collection of sets $g\cap N$ where $N$ is in $U$ is a monotonic collection of closed and bounded subsets of the plane and must have a common part. Thus, the sets in $U$ have an intersection which is a blocking set of $f$ relative to $H\times K$. It follows that there exists a minimal blocking set $C$ of $f$ relative to $H\times K$.

Assume $(C)_X$ intersects more than one component of $H$. Let $s$ be an open segment contained in $R-H$ such that $(C)_X$ has a point to the left of $s$ and a point to the right of $s$. Denote by $N$ the part of $H$ to the left of $s$ and by $M$ the part of $H$ to the right of $s$. By the minimality of $C$, there exist continuous functions $g$ and $h$, each of which has domain $H$ and range a subset of $K$, such that $g$ contains no point of $C_N$ and $h$ contains no point of $C_M$. Then $g_N\cup h_M$ is a continuous function which has domain $H$, range a subset of $K$ and contains no point of $C$, a contradiction.

Assume that $K$ is connected but that $\mathrm{Cl}((C)_X)$ is not. Let $s=(m, n)$ be a segment in $R-\mathrm{Cl}((C)_X)$ such that $(C)_X$ has a point on either side of $s$. As in the previous paragraph, there exist continuous functions $g$ and $h$ such that $(g)_X=H\cap(-\infty, m]$, $(h)_X=H\cap[n, \infty)$, $g\subset H\times K$, $h\subset H\times K$, and neither of $g$ and $h$ intersects $C$. Denote by $L$ the line segment with endpoints $(m, g(m))$ and $(n, h(n))$. Since $K$ is connected $(L)_Y\subset K$. Thus $g\cup L\cup h$ is a continuous function with domain $H$, range $K$ and contains no point of $C$, a contradiction.

NOTE. For the set $C$ defined in Theorem 1, $(C)_X$ is clearly non-degenerate. Thus, if $f$ is a real function with domain a nowhere-dense bounded closed set $H$, each component of $H$ is degenerate and $f$ is almost continuous.

We now give an example of an almost continuous function which is a $G_\delta$ set but which is not of Baire Class 1.

EXAMPLE 1. Denote by $K$ the Cantor middle-thirds set in $I=[0, 1]$ and by $J=(e_1, e_2, e_3, \cdots)$ the set of endpoints of complementary segments of $K$. Suppose $f$ is the function with domain $I$ and range $[-1, 1]$ such that:

(1) If $z$ is in $I-K$ and $(m, n)$ is the component of $I-K$ containing $z$, then $f(z)=\sin(1/(m-z)(n-z))$,

(2) if $z$ is in $K-J$, $f(z)=1$,

(3) if $z$ is in $J$, $f(z)=1/r$, where $z$ is $e_r$.

Since $f_K$ is totally discontinuous, $f$ is not of Baire Class 1. Clearly $f_{I-K}$ and $f_{K-J}$ are $G_\delta$ sets. Since no point of $f_J$ is a limit point of $f_J$, $f_J$ is a $G_\delta$ set [**5**, p. 68], so $f$ is a $G_\delta$ set.

Now, assume that $f$ is not almost continuous. By Theorem 1, there exists a minimal blocking set $C$ of $f$ relative to $I\times[-1, 1]$ and $(C)_X$ is connected and nondegenerate. If $s$ is a complementary segment of $K$, $f_s$ is continuous. Clearly, $(C)_X$ contains a complementary segment $s=(m, n)$ of $K$. By the minimality of $C$, there exist continuous functions $g$ and $h$, each of which has range a subset of $I$, neither of which intersects $C$, such that $(g)_X=[0, m]$ and $(h)_X=[n, 1]$.

Each of $(m, g(m))$ and $(n, h(n))$ is a limit point of $f_s$. Denote by each of $L_1$ and $L_2$ a line segment which contains no point of $C$ such that $L_1$ has endpoints $(m, g(m))$ and $(z_1, f(z_1))$ and $L_2$ has endpoints $(n, h(n))$ and $(z_2, f(z_2))$ where $m<z_1<z_2<n$. Then

$$g \cup L_1 \cup f_{[z_1, z_2]} \cup L_2 \cup h$$

is a continuous function with domain $I$, range a subset of $[-1, 1]$ and contains no point of $C$, a contradiction. Therefore $f$ is almost continuous.

The following example shows there exists an almost continuous function dense in $I^2$.

EXAMPLE 2. Denote by $U$ the set of all upper semicontinuous functions with domain $I$ and range a subset of $I$. Then $U$ is equally numerous with $R$ [**3**, p. 140]. Denote by $V$ a collection of mutually exclusive dense subsets of $I$ whose union is $I$ such that $V$ is equally numerous with $R$. Let $T$ be a reversible transformation with domain $V$ and range $U$.

Suppose $f$ is the function with range and domain $I$ such that $f(z)=g(z)$ where $g=T(N)$ and $N$ is the member of $V$ containing $z$. Assume $f$ is not almost continuous. Then there exists a minimal blocking set $C$ of $f$ relative to $I^2$, and $(C)_X$ is connected and nondegenerate. Then the function $u$ such that $u(z)=\max(C_z)_Y$ is an upper semicontinuous function with domain $(C)_X$. Thus, $u$ and $f$ intersect. Then $f$ and $C$ intersect, a contradiction. Therefore, $f$ is almost continuous, and clearly, is dense in the unit square.

## BIBLIOGRAPHY

**1.** Jack B. Brown, *Connectivity, semi-continuity, and the Darboux property*, Duke Math. J. **36** (1969), 559–562. MR **39** #7568.

**2.** J. L. Cornette, *Connectivity functions and images on Peano continua*, Fund. Math. **58** (1966), 183–192. MR **33** #6600.

**3.** C. Goffman, *Real functions*, Rinehart, New York, 1953. MR **14,** 855.

**4.** F. B. Jones and E. S. Thomas, Jr., *Connected $G_\delta$-graphs*, Duke Math. J. **33** (1966), 341–345. MR **33** #702.

**5.** R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R.I., 1962. MR **27** #709.

**6.** J. H. Roberts, *Zero-dimensional sets blocking connectivity functions*, Fund. Math. **57** (1965), 173–179. MR **33** #3270.

**7.** J. R. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263. MR **22** #8485.

Department of Mathematics, University of Alabama, University, Alabama 35486

*Current address* (K. R. Kellum): Department of Mathematics, Miles College, Birmingham, Alabama 35208