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A SPHERICAL SURFACE MEASURE INEQUALITY FOR CONVEX SETS

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ABSTRACT. Let the set C in the Euclidean space of n dimensions be closed, symmetric under reflection in the origin, and convex. The portion of the surface of the unit ball lying in C is shown to decrease in (the uniform) surface measure when C is replaced by AC, the image of C under any linear transformation A with norm no greater than one. Some cases of equality are discussed, and an application is given.

1. **Introduction.** We prove that the hypersurface area of the intersection of a convex symmetric set $C\subseteq E^n$ $(C=-C)$ with the unit sphere S decreases when we shrink C. Precisely, let μ be the uniform surface measure on S and let $A:E^n\to E^n$ be a linear transformation of norm $\leqq 1$. (The norm of A, denoted $\|A\|$, is given by $\|A\|=\sup\{|Ax|:x\in S\}$, and $|y|^2=\sum y_i^2$.)

THEOREM 1. *If C is closed, convex, and symmetric about* 0, *then* $\mu(AC\cap S)\leqq\mu(C\cap S)$.

A simple geometric argument proves Theorem 1 in the two-dimensional case. First let C be an infinite strip L meeting S in $I\cup(-I)$, the union of a symmetric pair of arcs (see Figure 1). Since the width of AL is no greater than that of L, $\mu(AL\cap S)\leqq\mu(L\cap S)$.

Next assume C is a polygon. Then $C\cap S$ may be expressed as the union of a finite collection of symmetric pairs of arcs I_j, $-I_j$. For each j let L_j be the symmetric strip such that $L_j\cap S=I_j\cup(-I_j)$ (see Figure 2). Since $\|A\|\leqq 1$, the part of C contained in the open unit ball B plays no role (that is, $AC\cap S=A(C\sim B)\cap S$). The remainder of C is contained in the union

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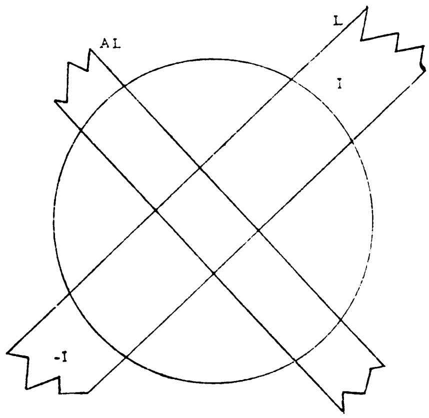

FIGURE 1

of the strips $L_j((C \sim B) \subseteq \bigcup L_j)$. Therefore, $AC \cap S \subseteq \bigcup (AL_j \cap S)$ so

$$\mu(AC \cap S) \leqq \mu(\bigcup (AL_j \cap S)) \leqq \sum_j \mu(AL_j \cap S) \leqq \sum \mu(L_j \cap S)$$
$$= \sum \mu(I_j \cup (-I_j)) = \mu(C \cap S).$$

In the general case, a bounded C is the intersection of a decreasing sequence of convex symmetric polygons P_n. Thus $\mu(AC \cap S) \leqq \mu(AP_n \cap S) \leqq \mu(P_n \cap S) \to \mu(C \cap S)$ as $n \to \infty$. If C is unbounded, C must be a line, a strip or the whole plane.

In the case $n \geqq 3$ we could not find a geometric argument. Indeed, for $n=3$ the convex symmetric hull of a triangle lying outside S offers difficulty. (See [2, Example (D), p. 407].)

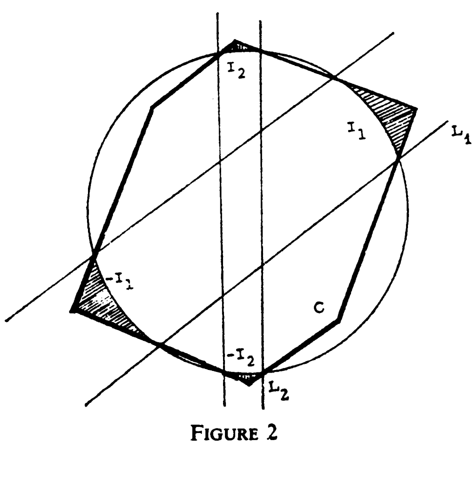

FIGURE 2

Theorem 1 is false if symmetry is dropped, if "convex" is replaced by "star-shaped", or if S is replaced by the surface of an ellipsoid.

The result is strengthened in Theorem 2 and the case of equality is discussed. An application to probability distributions is given in §4, extending results appearing in [1], [2], and [3].

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2. **Proof of Theorem 1.** We make use of the following corollary of a theorem of Anderson [1] (see Sherman's interesting paper [4]). Let $f_1, f_2 \geqq 0$ be even, bounded, and integrable on E^n, such that $\{x : f_i(x) \geqq \alpha\}$ is convex for each α. Then for $x \in E^n$, $f_1 * f_2(tx)$ is a nonincreasing function of $t > 0$.

We can take C compact. Since C is a decreasing limit of closed convex symmetric polyhedra, we can take C to be an arbitrary such polyhedron. Use orthogonal transformations to reduce A to a matrix $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, $0 \leqq \lambda_i \leqq 1$. To show that $\mu(DC \cap S)$ is a nondecreasing function of each λ_i it suffices to show that

$$u(\lambda) \equiv \mu(D_\lambda C \cap S) = \int_S \chi_C(D_\lambda^{-1} x)\, d\mu(x)$$

is a nondecreasing function of λ $(0 < \lambda \leqq 1)$ where $D_\lambda = \operatorname{diag}(\lambda, 1, \cdots, 1)$, and χ_C is the indicator function of C.

For $\varepsilon > 0$ let f_ε be the (C^∞) approximate identity

$$f_\varepsilon(x) = (2\pi\varepsilon)^{-n/2} \exp[-|x|^2/2\varepsilon];$$

set $\varphi_\varepsilon = \chi_C * f_\varepsilon$ and

$$u_\varepsilon(\lambda) = \int_S \varphi_\varepsilon(D_\lambda^{-1} x)\, d\mu(x).$$

Note that φ_ε is C^∞ and $\varphi_\varepsilon(x) \to \chi_C(x)$ as $\varepsilon \to 0$ (unless $x \in \partial C$; but $\mu(\partial C \cap S) = 0$ as C is a polyhedron) so $u_\varepsilon(\lambda) \to u(\lambda)$. Therefore it suffices to show that $u_\varepsilon'(\lambda) \geqq 0$. We put $\psi(x) = \varphi_\varepsilon(D_\lambda^{-1} x)$ and write

$$u_\varepsilon'(\lambda) = -\lambda^{-1} \int_S x_1 \frac{\partial \psi}{\partial x_1}\, d\mu(x).$$

Consider now $\chi_B * \psi$, where B is the unit ball. It follows from the corollary to Anderson's theorem that $\chi_B * \psi$ has its maximum at the origin. Hence, with $z = (1, 0, \cdots, 0)$,

$$0 \geqq \frac{d^2}{dt^2} [(\chi_B * \psi)(tz)]_{t=0} = \frac{\partial^2}{\partial x_1^2} (\chi_B * \psi)\big|_{x=0}.$$

Write $x=(x_1, x')$ where $x'=(x_2, \cdots, x_n)\in E^{n-1}$ and let $a(x')=(1-|x'|^2)^{1/2}$. Then

$$\begin{aligned} 0 &\geqq \int_B \frac{\partial^2\psi}{\partial x_1^2}\,dx \\ &= \int_{|x'|\leqq 1}\left[\int_{-a(x')}^{a(x')} \frac{\partial^2\psi(x_1, x')}{\partial^2 x_1}\,dx_1\right] dx' \\ &= 2\int_{|x'|\leqq 1} \frac{\partial\psi(a(x'), x')}{\partial x_1}\,dx' \\ &= \int_S x_1 \frac{\partial\psi}{\partial x_1}\,d\mu(x) = -\lambda u_\varepsilon'(\lambda). \quad \text{Q.E.D.} \end{aligned}$$

3. **A stronger result and some cases of equality.** Theorem 1 implies that for a (not necessarily closed) convex symmetric set C,

$$\mu(\bar{C} \cap S) = \mu(C^0 \cap S) + \mu(\partial C \cap S) \geqq \mu(A\bar{C} \cap S).$$

In Theorem 2 we show that $\mu(\partial C\cap S)$ is not needed for nontrivial A (even though $\mu(C^0\cap S)$ may be 0).

THEOREM 2. *If* $\|A\|\leqq 1$ *but* A *is not orthogonal, then*

$$\mu(A\bar{C} \cap S) \leqq \mu(C^0 \cap S). \tag{1}$$

PROOF. Again we can take $A=D_\lambda=\text{diag}(\lambda, 1, \cdots, 1)$ where now $0<\lambda<1$. For $\lambda<t<1$, Theorem 1 gives

$$\mu(D_\lambda\bar{C} \cap S) \leqq \mu(D_t\bar{C} \cap S) = \int_S \chi_t(x)\,d\mu(x)$$

where χ_t denotes the indicator function of $D_t\bar{C}$. We show that as $t\nearrow 1$, $\chi_t \rightarrow \chi_{C^0\cap S}$ almost everywhere in S. By symmetry restrict attention to the half-space $H=\{x|x_1>0\}$. There are four cases. If (i) $x\in C^0\cap H$ then $\chi_t(x)\rightarrow 1$ as $t\nearrow 1$. If (ii) $x\in\partial C\cap H$ but $D_a x\notin\bar{C}$ for $a>1$ (i.e., x belongs to the "upper boundary" of C) then $\chi_{C^0\cap S}(x)=0=\chi_t(x)$ for $t<1$. Two cases remain: (iii) $x\in\partial C\cap H$ and $D_{1+\varepsilon}x\in\partial C$ for small $\varepsilon>0$ (i.e., $x\in H\cap L_1$, where L_1 is the "lateral boundary" of C) and (iv) $x\in\partial C\cap H$ and $D_{1+\varepsilon}x\in C^0$ for small $\varepsilon>0$ (i.e., $x\in H\cap L_2$, where L_2 is the "lower boundary" of C). In (iii) the projection of L_1 into $H_0=\{x|x_1=0\}$ lies in the boundary of the projection of C into H_0, so $\mu(H\cap L_1\cap S)=0$. Finally, $\mu(H\cap L_2\cap S)=0$ by the following lemma, since L_2 is the graph of a convex function.

LEMMA. *If* $G\subset E^n$ *is the graph of a nonnegative convex function then* $\mu(G\cap S)=0$.

PROOF. Induction. The case $n=2$ is obvious. For $n\geqq 3$ first integrate over sections parallel to the "vertical" axis; each such integral is zero by induction. Q.E.D.

The case of equality in (1) has not been settled. In two dimensions it is easy to show from the geometric argument that strict inequality holds unless $A\bar{C}\cap S=LC^0\cap S$ up to a set of μ-measure 0, for some orthogonal L.

In the special case $\|A\|<1$, equality holds in (1) if and only if

$$C^0\cap S=\varnothing \quad \text{or} \quad A\bar{C}\cap S=S. \tag{2}$$

Sufficiency is clear. If (2) fails then $\mu(C^0\cap S)>0$ and $\mu(A\bar{C}\cap S)<\mu(S)$, so we can assume that $C^0\cap S\neq S$. We can also take $A=\operatorname{diag}(d_1,\cdots,d_n)=D$ with $0\leqq d_i<1$; let $d=\max\{d_i\}<1$. By Theorem 2, $\mu(D\bar{C}\cap S)\leqq \mu(d\bar{C}\cap S)\leqq\mu(C^0\cap S)$ so it suffices to show that $\mu(d\bar{C}\cap S)<\mu(C^0\cap S)$. Let $x\in d\bar{C}\cap S$. Then $y=d^{-1}x\in\bar{C}$ so $x\in[0,y)\subset C^0$. Thus $d\bar{C}\cap S$ (closed in S) is contained in $C^0\cap S$ (open in S). Since $C^0\cap S$ is a proper subset of S this containment must be strict.

5. **Extension to radial measures.** Let ν be a radial Borel measure on E^n. That is, $\nu(E)=\nu(LE)$ for every Borel set E and orthogonal transformation L. For any positive definite $\Sigma:E^n\to E^n$ define $\nu_\Sigma(E)=\nu(\Sigma^{-1/2}E)$ ($\Sigma^{1/2}$ can be any square root of Σ since ν is radial). Theorem 2 easily gives the following result.

THEOREM 3. *If C is convex, symmetric, and if $\Sigma_2-\Sigma_1$ is positive semidefinite with $\Sigma_1\neq\Sigma_2$ then $\nu_{\Sigma_2}(\bar{C})\leqq\nu_{\Sigma_1}(C^0)$.*

If ν is a probability measure this result has the following interpretation. The family $\{\nu_\Sigma\}$ is a generalized scale parameter family of symmetric distributions. (If ν has finite second moments then the covariance matrix of ν_Σ is proportional to Σ.) Theorem 3 says that as the scale parameter (or covariance matrix) Σ decreases, the distribution ν_Σ becomes more concentrated, or "peaked", about the origin in the sense of Sherman [**4**], i.e., the probability of every convex symmetric set is increased. This was proved for the Gaussian distribution by Anderson [**1**, Corollary 3, p. 173]. Also see Dudley, Feldman and LeCam, [**2**, (A), p. 406], and Gross [**3**, Lemma 5.2, p. 384]. The case of the uniform distribution on the unit ball is proved by Gross [**3**, Lemma 5.1, p. 383].

REFERENCES

1. T. W. Anderson, *The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. **6** (1955), 170–176. MR **16**, 1005.

2. R. M. Dudley, J. Feldman and L. LeCam, *On seminorms and probabilities, and abstract Wiener spaces*, Ann. of Math. (2) **93** (1971), 390–408.

3. L. Gross, *Measurable functions on Hilbert space*, Trans. Amer. Math. Soc. **105** (1962), 372–390. MR **26** #5121.

4. S. Sherman, *A theorem on convex sets with applications*, Ann. Math. Statist. **26** (1955), 763–767. MR **17**, 655.

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