A BAIRE SPACE EXTENSION

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Abstract. A characterization of Baire spaces is given. Using this characterization, it is shown that every topological space is a dense subspace of some compact Baire space.

If $X$ and $Y$ are topological spaces and if $X$ is a dense subspace of $Y$, then $Y$ will be called an extension of $X$. If $Y$ has property $P$, then it will be called a $P$ extension of $X$. Many people have investigated, for various nonhereditary properties $P$, the classes of spaces having $P$ extensions. For example, every Tychonoff space has a compact, Hausdorff extension, and every Hausdorff space has an $H$-closed extension [3]. An $H$-closed (or absolutely closed) space is a Hausdorff space such that every open filter base has a cluster point. Since a compact Hausdorff space is a Baire space, then every Tychonoff space has a Tychonoff, Baire space extension. A Baire space is a space such that every nonempty open subset is of second category in the space. Herrlich gave an example in [2] of an $H$-closed space (in fact a minimal Hausdorff space) which is of first category in itself and hence not a Baire space. Therefore there exist Hausdorff spaces which have no Hausdorff Baire space extensions. In the Corollary to Theorem 3, we see that every topological space has a compact Baire space extension.

We shall use the following characterization of Baire spaces, which is similar to a characterization given in [1] of spaces of second category in themselves.

Theorem 1. $X$ is a Baire space if and only if every countable point finite open cover of $X$ is locally finite at a dense set of points.

Proof. Suppose that $X$ is a Baire space. Let $\mathcal{U}=\{U_i\}$ be a countable point finite open cover of $X$, and let $U$ be a nonempty open subset of $X$. Then $U$ is of second category in $X$. Assume that $\mathcal{U}$ is not locally finite at any point of $U$. Let $V=\{V_i\}$, where each $V_i=U_i\cap U$. Then each open set in $U$ intersects infinitely many members of $V$. Let $\mathcal{N}=\{N_i|\mathcal{N}$$\subseteq\omega$ and $\omega \setminus \mathcal{N}$ is finite$\}$, which is countable. Let $\{N_i|i\in\omega\}$ be a well ordering of $\mathcal{N}$. For each $i$, define $X_i=\text{Bd}\big]\{V_j|j\in N_i\}\}$. Each $X_i$ is closed and $\text{Int} X_i=\emptyset$, so that each $X_i$ is...
nowhere dense. Let \( x \in U \). There exists an integer \( k \) such that \( x \) is in the members of \( \{ V_j \}_{j \in \mathbb{N}_k} \), but not in the other members of \( \mathcal{V} \). Let \( V \) be an open set containing \( x \). Then \( V \) intersects some members of \( \{ V_j \}_{j \in \mathbb{N}_k} \). But since \( x \in \bigcup \{ V_j \}_{j \in \mathbb{N}_k} \), then \( x \in X_k \). This says that \( U = \bigcup_{i=1}^{\infty} U \cap X_i \), which contradicts \( U \) being of second category in \( X \).

Conversely, suppose that every countable point finite open cover of \( X \) is locally finite at a dense set of points. Let \( U \) be a nonempty open subset of \( X \). Assume that \( U \) is of first category in \( X \). Then \( U = \bigcup_{i=1}^{\infty} X_i \), where Int \( X_i = \emptyset \) for each \( i \). Let \( U_0 = X \). and for each \( i \), let \( U_i = U \cap X_i \). Let \( \mathcal{U} = \{ U_0, U_1, \ldots \} \), which is a countable point finite open cover of \( X \). Then there exists an \( x \in U \) such that \( \mathcal{U} \) is locally finite at \( x \). So there is an open set \( V \) such that \( x \in V \subset U \) and \( V \) intersects only finitely many members of \( \mathcal{U} \). But \( V \) is not contained in \( \bigcup_{i=1}^{\infty} X_i \) for each \( i \). Then \( V \) must intersect every member of \( \mathcal{U} \), which is a contradiction.

**Lemma.** Let \( \mathcal{B} \) be a base for the topological space \( X \). If \( X \) is not a Baire space, then there exist a countable point finite collection \( \{ U_i \} \) of open subsets of \( X \) and a countable collection \( \{ B_i \} \) such that \( B_k \subseteq B_n \cap \bigcap_{i=1}^{n} U_i \) for every \( k \).

**Proof.** By Theorem 1, there exist a countable point finite open cover \( \mathcal{V} = \{ V_i \} \) of \( X \) and an open subset \( V \) of \( X \) such that \( \mathcal{V} \) is not locally finite at any point of \( V \). Let \( B_i \in \mathcal{B} \) be contained in \( V \). There exists an integer \( i_1 \) such that \( B_i \cap V_i \neq \emptyset \). Let \( B_2 \in \mathcal{B} \) be contained in \( B_i \cap V_i \). Proceeding by induction, suppose we have defined distinct \( \{ i_1, \ldots, i_{n-1} \} \) and \( \{ B_1, \ldots, B_n \} \subseteq \mathcal{B} \) such that \( B_k \cap B_{k+1} \cap \bigcap_{i=1}^{k} V_i \) for every \( 2 \leq k \leq n \). Then since \( \mathcal{V} \) is not locally finite in \( V \), there exists an integer \( i_n \) distinct from the elements of \( \{ i_1, \ldots, i_{n-1} \} \) such that \( B_i \cap V_{i_n} \neq \emptyset \). Let \( B_{i+1} \in \mathcal{B} \) be contained in \( B_{i_n} \cap V_{i_n} \). Thus \( B_{i+1} \subseteq B_{i_n} \cap \bigcap_{i=1}^{i_n} V_i \), so that \( \{ B_i \} \) is therefore defined by induction. For each \( j \), take \( U_j = V_{i_j} \). Then \( \{ U_j \} \) and \( \{ B_i \} \) satisfy the conclusion of the Lemma.

An *open filter* \( \mathcal{F} \) on a space \( X \) is a nonempty collection of nonempty open subsets of \( X \) satisfying:

(a) If \( U, V \in \mathcal{F} \), then \( U \cap V \in \mathcal{F} \).

(b) If \( U \in \mathcal{F} \) and \( V \) is an open set containing \( U \), then \( V \in \mathcal{F} \). An *open ultrafilter* is an open filter which is maximal in the collection of open filters. An open filter \( \mathcal{F} \) is free if \( \bigcap \mathcal{F} = \emptyset \).

**Theorem 2.** If \( X \) is not a Baire space, then \( X \) has a free open ultrafilter.

**Proof.** Let \( \{ U_i \} \) and \( \{ B_i \} \) be defined as in the Lemma. Then \( \{ B_i \} \) is an open filter base on \( X \). Let \( \mathcal{U} \) be an open ultrafilter on \( X \) containing \( \{ B_i \} \). Since \( \bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n} U_i \) and \( \{ U_i \} \) is point finite, then \( \bigcap_{i=1}^{n} B_i = \emptyset \). Therefore \( \bigcap \mathcal{U} = \emptyset \), so that \( \mathcal{U} \) is free.
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The following construction is similar to the Katetov extension found for example in [3] and [4].

Let \( X \) be a topological space, and let \( F \) be a set of open filters on \( X \). Let \( X_F \) be the disjoint union of \( X \) and \( F \). For each open \( U \) in \( X \), let \( U^* = U \cup \{ F \in F | U \in F \} \). Note that \( (U \cap V)^* = U^* \cap V^* \) for every open \( U \) and \( V \) in \( X \), and \( \phi^* = \emptyset \). Let \( X_F \) have the topology generated by the base \( \mathcal{B} = \{ U^* | U \text{ is open in } X \} \). Clearly \( X \) is a dense subspace of \( X_F \).

**Theorem 3.** If \( F \) is any of the following sets,

(a) all open filters on \( X \),
(b) all open ultrafilters on \( X \),
(c) all free open ultrafilters on \( X \),

then \( X_F \) is a Baire space.

**Proof.** We shall prove Theorem 3 for case (c). Assume that \( X_F \) is not a Baire space. Let \( \{ U_i \} \) and \( \{ B_i \} \) be defined as in the Lemma for the space \( X_F \). Each \( B_i = V_i^* \) for some open \( V_i \) in \( X \). Let \( \mathcal{U} \) be an open ultrafilter on \( X \) containing \( \{ V_i \} \). Since \( \bigcap_{i=1}^{\infty} V_i \subseteq \bigcap_{i=1}^{\infty} B_i \subseteq \bigcap_{i=1}^{\infty} U_i \) and \( \{ U_i \} \) is point finite, then \( \bigcap_{i=1}^{\infty} V_i = \emptyset \). Therefore \( \bigcap \mathcal{U} = \emptyset \), so that \( \mathcal{U} \) is free. Hence \( \mathcal{U} \in F \). Since each \( V_i \in \mathcal{U} \), then \( \mathcal{U} \in V_i^* \). Thus \( \mathcal{U} \in \bigcap_{i=1}^{\infty} B_i \subseteq \bigcap_{i=1}^{\infty} U_i = \emptyset \). This contradiction establishes that \( X_F \) is a Baire space.

**Corollary.** Every topological space is a dense subspace of some compact Baire space.

**Proof.** If \( X \) is a topological space, let \( X_F^* \) be the one-point compactification of \( X_F \), where \( F \) is any of the sets given in the statement of Theorem 3. Since \( X_F \) is a dense open subspace of \( X_F^* \), and \( X_F^* \) is a Baire space, then \( X_F^* \) must be a Baire space.

**Theorem 4.** Let \( F \) be the set of all free open ultrafilters on \( X \), and let \( f \) be a continuous function from \( X \) into a \( T_3 \)-space \( Y \) such that \( f(X) \) is dense in \( Y \). Then there exists a subspace \( Z \) of \( Y_F \) containing \( X \) and a continuous function \( g \) from \( Z \) onto \( Y \) such that \( g|_X = f \) and \( g|_Z : X \) is a homeomorphism from \( Z \setminus X \) onto \( Y \setminus f(X) \). Furthermore, if \( f \) is a homeomorphism then \( g \) is a homeomorphism.

**Proof.** For each \( y \in Y \setminus f(X) \), let \( V_y \) be the open filter on \( Y \) consisting of all open subsets of \( Y \) containing \( y \). Let \( \mathcal{U}_y \) be an open ultrafilter on \( X \) generated by \( f^{-1}(V_y) \). \( \mathcal{U}_y \) is free since if \( x \in X \), then there exists an open set \( V \) in \( Y \) with \( y \in V \) and \( f(x) \notin V \). Since \( V \in \mathcal{U}_y \), \( f^{-1}(V) \in \mathcal{U}_y \). But \( x \notin f^{-1}(V) \), so that \( \mathcal{U}_y \) must be free. Let \( Z = X \cup \{ \mathcal{U}_y | y \in Y \setminus f(X) \} \). Define \( g: Z \to Y \) by \( g(x) = f(x) \) if \( x \in X \) and \( g(\mathcal{U}_y) = y \) for each \( y \in Y \setminus f(X) \). To see that \( g \) is well defined, let \( y_1 \) and \( y_2 \) be distinct points of \( Y \setminus f(X) \). Then there exists an open \( U \) in \( Y \) such that \( y_1 \notin U \) and \( y_2 \notin U \). Hence \( U \in \mathcal{U}_{y_1} \) and \( Y \setminus U \in \mathcal{U}_{y_2} \). But then \( f^{-1}(U) \in \mathcal{U}_{y_1} \) and \( X \setminus f^{-1}(U) \in \mathcal{U}_{y_2} \). Therefore \( \mathcal{U}_{y_1} \neq \mathcal{U}_{y_2} \).
The fact that $g$ is continuous follows from the fact that $X$ is dense in $Z$, the fact that $Y$ is regular, and the fact that for each $z \in Z$ the restriction $g|X \cup \{z\}$ is continuous.

To see that $g|Z \times X$ is a homeomorphism, let $h = g|Z \times X$. Let $y \in Y \setminus X$, and let $U^*$ contain $h^{-1}(y)$ for some open $U$ in $X$. Then $h^{-1}(y) = \mathcal{U}_y$ and $U \in \mathcal{U}_y$. Hence $f^{-1}(V) \subseteq U$ for some $V \in \mathcal{V}_y$. Let $y' \in V$. Then $V \in \mathcal{V}_y$, so that $f^{-1}(V) \subseteq \mathcal{U}_y$. But then $U \in \mathcal{U}_y$, so that $h^{-1}(y') = \mathcal{U}_y \subseteq U^*$. Therefore $h^{-1}(V) \subseteq U^*$.

Finally, suppose that $f$ is a homeomorphism. Let $y \in Y$, and let $U$ be open in $X$ such that $g^{-1}(y) \subseteq U^*$. If $y \in f(X)$, then there exists an open $V$ in $Y$ such that $f^{-1}(V) \subseteq U$. If $y \notin f(X)$, then $g^{-1}(y) = \emptyset$, so that $U \in \mathcal{U}_y$. But then there exists an open subset $V$ of $Y$ containing $y$ such that $f^{-1}(V) \subseteq U$. In either case, let $y' \in V$. If $y' \in f(X)$, then $g^{-1}(y') = f^{-1}(y') \subseteq f^{-1}(V) \subseteq U \subseteq U^*$. If $y' \notin f(X)$, then $V \in \mathcal{V}_y$, so that $f^{-1}(V) \subseteq \mathcal{U}_y$. Hence $U \in \mathcal{U}_y$, so that $g^{-1}(y') = \mathcal{U}_y \subseteq U^*$. Therefore, since $y'$ is arbitrary, $g^{-1}(V) \subseteq U^*$.

**Corollary 1.** If $X$ is a $T_3$-space, then every $T_3$-space extension of $X$ can be embedded as a subspace of $X_F$ containing $X$.

**Corollary 2.** Let $X$ be a Tychonoff space and $Y$ be a Hausdorff space. Then $Y$ is a compactification of $X$ if and only if $Y$ can be embedded as a compact subspace of $X_F$ containing $X$.

We might note that it follows from Corollary 2 that $X_F$ is in general not Hausdorff. In fact if $X$ is $H$-closed, no subset of $X_F$ containing $X$ as a proper subset is ever Hausdorff. This is because a Hausdorff space $X$ is $H$-closed if and only if every embedding of $X$ into a Hausdorff space is a closed embedding. Therefore if $X$ is an $H$-closed space which is not a Baire space (e.g., Herrlich’s example in [2]), then $X$ is a Hausdorff space which cannot have a Hausdorff Baire space extension. Observe that $X$ could never be regular, since a regular $H$-closed space must be compact, and hence a Baire space. It would be of interest to know whether every regular space has a regular Baire space extension.

**References**


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