PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

A BAIRE SPACE EXTENSION

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ABSTRACT. A characterization of Baire spaces is given. Using this characterization, it is shown that every topological space is a dense subspace of some compact Baire space.

If X and Y are topological spaces and if X is a dense subspace of Y, then Y will be called an extension of X. If Y has property P, then it will be called a P extension of X. Many people have investigated, for various nonhereditary properties P, the classes of spaces having P extensions. For example, every Tychonoff space has a compact, Hausdorff extension, and every Hausdorff space has an H-closed extension [3]. An H-closed (or absolutely closed) space is a Hausdorff space such that every open filter base has a cluster point. Since a compact Hausdorff space is a Baire space, then every Tychonoff space has a Tychonoff, Baire space extension. A Baire space is a space such that every nonempty open subset is of second category in the space. Herrlich gave an example in [2] of an H-closed space (in fact a minimal Hausdorff space) which is of first category in itself and hence not a Baire space. Therefore there exist Hausdorff spaces which have no Hausdorff Baire space extensions. In the Corollary to Theorem 3, we see that every topological space has a compact Baire space extension.

We shall use the following characterization of Baire spaces, which is similar to a characterization given in [1] of spaces of second category in themselves.

THEOREM 1. *X is a Baire space if and only if every countable point finite open cover of X is locally finite at a dense set of points.*

PROOF. Suppose that X is a Baire space. Let $\mathscr{U}=\{U_i\}$ be a countable point finite open cover of X, and let U be a nonempty open subset of X. Then U is of second category in X. Assume that $\mathscr{U}$ is not locally finite at any point of U. Let $\mathscr{V}=\{V_i\}$, where each $V_i=U_i\cap U$. Then each open set in U intersects infinitely many members of $\mathscr{V}$. Let ω be the set of natural numbers, and let $\mathscr{N}=\{N|N\subset\omega$ and $\omega\backslash N$ is finite$\}$, which is countable. Let $\{N_i|i\in\omega\}$ be a well ordering of $\mathscr{N}$. For each i, define $X_i=\mathrm{Bd}[\bigcup\{V_j|j\in N_i\}]$. Each X_i is closed and $\mathrm{Int}\, X_i=\varnothing$, so that each X_i is

Presented to the Society, September 3, 1971; received by the editors April 15, 1971.

AMS 1970 *subject classifications*. Primary 54C25, 54D25, 54D30, 54D35.

Key words and phrases. Baire space, H-closed space, open filter, open ultrafilter.

nowhere dense. Let $x \in U$. There exists an integer k such that x is in the members of $\{V_i | i \in \omega \backslash N_k\}$, but not in the other members of $\mathscr{V}$. Let V be an open set containing x. Then V intersects some members of $\{V_j | j \in N_k\}$. But since $x \notin \bigcup \{V_j | j \in N_k\}$, then $x \in X_k$. This says that $U = \bigcup_{i=1}^{\infty} U \cap X_i$, which contradicts U being of second category in X.

Conversely, suppose that every countable point finite open cover of X is locally finite at a dense set of points. Let U be a nonempty open subset of X. Assume that U is of first category in X. Then $U = \bigcup_{i=1}^{\infty} X_i$, where $\operatorname{Int} \bar{X}_i = \varnothing$ for each i. Let $U_0 = X$, and for each i, let $U_i = U \backslash \bigcup_{j=1}^{i} \bar{X}_j$. Let $\mathscr{U} = \{U_0, U_1, \cdots\}$, which is a countable point finite open cover of X. Then there exists an $x \in U$ such that $\mathscr{U}$ is locally finite at x. So there is an open set V such that $x \in V \subset U$ and V intersects only finitely many members of $\mathscr{U}$. But V is not contained in $\bigcup_{j=1}^{i} \bar{X}_j$ for each i. Then V must intersect every member of $\mathscr{U}$, which is a contradiction.

LEMMA. *Let $\mathscr{B}$ be a base for the topological space X. If X is not a Baire space, then there exist a countable point finite collection $\{U_i\}$ of open subsets of X and a countable collection $\{B_i\} \subset \mathscr{B}$ such that $B_{k+1} \subset B_k \cap (\bigcap_{i=1}^{k} U_i)$ for every k.*

PROOF. By Theorem 1, there exist a countable point finite open cover $\mathscr{V} = \{V_i\}$ of X and an open subset V of X such that $\mathscr{V}$ is not locally finite at any point of V. Let $B_1 \in \mathscr{B}$ be contained in V. There exists an integer i_1 such that $B_1 \cap V_{i_1} \neq \varnothing$. Let $B_2 \in \mathscr{B}$ be contained in $B_1 \cap V_{i_1}$. Proceeding by induction, suppose we have defined distinct $\{i_1, \cdots, i_{n-1}\}$ and $\{B_1, \cdots, B_n\} \subset \mathscr{B}$ such that $B_k \subset B_{k-1} \cap (\bigcap_{j=1}^{k-1} V_{i_j})$ for every $2 \leqq k \leqq n$. Then since $\mathscr{V}$ is not locally finite in V, there exists an integer i_n distinct from the elements of $\{i_1, \cdots, i_{n-1}\}$ such that $B_n \cap V_{i_n} \neq \varnothing$. Let $B_{n+1} \in \mathscr{B}$ be contained in $B_n \cap V_{i_n}$. Thus $B_{n+1} \subset B_n \cap (\bigcap_{j=1}^{n} V_{i_j})$, so that $\{B_i\}$ is therefore defined by induction. For each j, take $U_j = V_{i_j}$. Then $\{U_i\}$ and $\{B_i\}$ satisfy the conclusion of the Lemma.

An *open filter* $\mathscr{F}$ on a space X is a nonempty collection of nonempty open subsets of X satisfying:

(a) If $U, V \in \mathscr{F}$, then $U \cap V \in \mathscr{F}$.

(b) If $U \in \mathscr{F}$ and V is an open set containing U, then $V \in \mathscr{F}$. An *open ultrafilter* is an open filter which is maximal in the collection of open filters. An open filter $\mathscr{F}$ is *free* if $\bigcap \mathscr{F} = \varnothing$.

THEOREM 2. *If X is not a Baire space, then X has a free open ultrafilter.*

PROOF. Let $\{U_i\}$ and $\{B_i\}$ be defined as in the Lemma. Then $\{B_i\}$ is an open filter base on X. Let $\mathscr{U}$ be an open ultrafilter on X containing $\{B_i\}$. Since $\bigcap_{i=1}^{\infty} B_i \subset \bigcap_{i=1}^{\infty} U_i$ and $\{U_i\}$ is point finite, then $\bigcap_{i=1}^{\infty} B_i = \varnothing$. Therefore $\bigcap \mathscr{U} = \varnothing$, so that $\mathscr{U}$ is free.

The following construction is similar to the Katětov extension found for example in [3] and [4].

Let X be a topological space, and let F be a set of open filters on X. Let X_F be the disjoint union of X and F. For each open U in X, let $U^* = U \cup \{\mathscr{F} \in F \mid U \in \mathscr{F}\}$. Note that $(U \cap V)^* = U^* \cap V^*$ for every open U and V in X, and $\phi^* = \varnothing$. Let X_F have the topology generated by the base $\mathscr{B} = \{U^* \mid U \text{ is open in } X\}$. Clearly X is a dense subspace of X_F.

THEOREM 3. *If F is any of the following sets,*

(a) *all open filters on X,*

(b) *all open ultrafilters on X,*

(c) *all free open ultrafilters on X,*

then X_F is a Baire space.

PROOF. We shall prove Theorem 3 for case (c). Assume that X_F is not a Baire space. Let $\{U_i\}$ and $\{B_i\}$ be defined as in the Lemma for the space X_F. Each $B_i = V_i^*$ for some open V_i in X. Let $\mathscr{U}$ be an open ultrafilter on X containing $\{V_i\}$. Since $\bigcap_{i=1}^{\infty} V_i \subset \bigcap_{i=1}^{\infty} B_i \subset \bigcap_{i=1}^{\infty} U_i$ and $\{U_i\}$ is point finite, then $\bigcap_{i=1}^{\infty} V_i = \varnothing$. Therefore $\bigcap \mathscr{U} = \varnothing$, so that $\mathscr{U}$ is free. Hence $\mathscr{U} \in F$. Since each $V_i \in \mathscr{U}$, then $\mathscr{U} \in V_i^* = B_i$. Thus $\mathscr{U} \in \bigcap_{i=1}^{\infty} B_i \subset \bigcap_{i=1}^{\infty} U_i = \varnothing$. This contradiction establishes that X_F is a Baire space.

COROLLARY. *Every topological space is a dense subspace of some compact Baire space.*

PROOF. If X is a topological space, let X_F^* be the one-point compactification of X_F, where F is any of the sets given in the statement of Theorem 3. Since X_F is a dense open subspace of X_F^*, and X_F is a Baire space, then X_F^* must be a Baire space.

THEOREM 4. *Let F be the set of all free open ultrafilters on X, and let f be a continuous function from X into a T_3-space Y such that $f(X)$ is dense in Y. Then there exists a subspace Z of Y_F containing X and a continuous function g from Z onto Y such that $g|_X = f$ and $g|_{Z\setminus X}$ is a homeomorphism from $Z \setminus X$ onto $Y \setminus f(X)$. Furthermore, if f is a homeomorphism then g is a homeomorphism.*

PROOF. For each $y \in Y \setminus f(X)$, let $\mathscr{V}_y$ be the open filter on Y consisting of all open subsets of Y containing y. Let $\mathscr{U}_y$ be an open ultrafilter on X generated by $f^{-1}(\mathscr{V}_y)$. $\mathscr{U}_y$ is free since if $x \in X$, then there exists an open set V in Y with $y \in V$ and $f(x) \notin V$. Since $V \in \mathscr{V}_y$, $f^{-1}(V) \in \mathscr{U}_y$. But $x \notin f^{-1}(V)$, so that $\mathscr{U}_y$ must be free. Let $Z = X \cup \{\mathscr{U}_y \mid y \in Y \setminus f(X)\}$. Define $g: Z \to Y$ by $g(x) = f(x)$ if $x \in X$ and $g(\mathscr{U}_y) = y$ for each $y \in Y \setminus f(X)$. To see that g is well defined, let y_1 and y_2 be distinct points of $Y \setminus f(X)$. Then there exists an open U in Y such that $y_1 \in U$ and $y_2 \notin \bar{U}$. Hence $U \in \mathscr{V}_{y_1}$ and $Y \setminus \bar{U} \in \mathscr{V}_{y_2}$. But then $f^{-1}(U) \in \mathscr{U}_{y_1}$ and $X \setminus f^{-1}(\bar{U}) \in \mathscr{U}_{y_2}$. Therefore $\mathscr{U}_{y_1} \neq \mathscr{U}_{y_2}$.

The fact that g is continuous follows from the fact that X is dense in Z, the fact that Y is regular, and the fact that for each $z\in Z$ the restriction $g|X\cup\{z\}$ is continuous.

To see that $g|_{Z\setminus X}$ is a homeomorphism, let $h=g|_{Z\setminus X}$. Let $y\in Y\setminus X$, and let U^* contain $h^{-1}(y)$ for some open U in X. Then $h^{-1}(y)=\mathscr{U}_y$ and $U\in\mathscr{U}_y$. Hence $f^{-1}(V)\subset U$ for some $V\in\mathscr{V}_y$. Let $y'\in V$. Then $V\in\mathscr{V}_{y'}$, so that $f^{-1}(V)\in\mathscr{U}_{y'}$. But then $U\in\mathscr{U}_{y'}$, so that $h^{-1}(y')=\mathscr{U}_{y'}\in U^*$. Therefore $h^{-1}(V)\subset U^*$.

Finally, suppose that f is a homeomorphism. Let $y\in Y$, and let U be open in X such that $g^{-1}(y)\in U^*$. If $y\in f(X)$, then there exists an open V in Y such that $f^{-1}(V)\subset U$. If $y\in Y\setminus f(X)$, then $g^{-1}(y)=\mathscr{U}_y\in U^*$, so that $U\in\mathscr{U}_y$. But then there exists an open subset V of Y containing y such that $f^{-1}(V)\subset U$. In either case, let $y'\in V$. If $y'\in f(X)$, then $g^{-1}(y')=f^{-1}(y')\in f^{-1}(V)\subset U\subset U^*$. If $y'\in Y\setminus f(X)$, then $V\in\mathscr{V}_{y'}$, so that $f^{-1}(V)\in\mathscr{U}_{y'}$. Hence $U\in\mathscr{U}_{y'}$, so that $g^{-1}(y')=\mathscr{U}_{y'}\in U^*$. Therefore, since y' is arbitrary, $g^{-1}(V)\subset U^*$.

COROLLARY 1. *If X is a T_3-space, then every T_3-space extension of X can be embedded as a subspace of X_F containing X.*

COROLLARY 2. *Let X be a Tychonoff space and Y be a Hausdorff space. Then Y is a compactification of X if and only if Y can be embedded as a compact subspace of X_F containing X.*

We might note that it follows from Corollary 2 that X_F is in general not Hausdorff. In fact if X is H-closed, no subset of X_F containing X as a proper subset is ever Hausdorff. This is because a Hausdorff space X is H-closed if and only if every embedding of X into a Hausdorff space is a closed embedding. Therefore if X is an H-closed space which is not a Baire space (e.g., Herrlich's example in [**2**]), then X is a Hausdorff space which cannot have a Hausdorff Baire space extension. Observe that X could never be regular, since a regular H-closed space must be compact, and hence a Baire space. It would be of interest to know whether every regular space has a regular Baire space extension.

REFERENCES

1. J. R. Fogelgren and R. A. McCoy, *Some topological properties defined by homeomorphism groups*, Arch. Math. **9** (1971), 1–6.

2. H. Herrlich, *Nicht alle T_2-minimalen Räume sind von 2. Kategorie*, Math. Z. **91** (1966), 185. MR **32** #6392.

3. M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis Pěst. Mat. Fys. **69** (1940), 36–49. MR **1**, 317.

4. C. T. Liu, *Absolutely closed spaces*, Trans. Amer. Math. Soc. **130** (1968), 86–104. MR **36** #2107.

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