FIBER HOMOTOPIC TRIVIAL BUNDLES OVER COMPLEX PROJECTIVE SPACES

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Abstract. We give a more illuminating derivation of a well-known condition (obtained by M. Atiyah and J. A. Todd) for certain vector bundles over complex projective spaces to be stably fiber homotopic trivial, together with a generalization.

Let $H$ be the Hopf line bundle over the complex projective space $CP^{k-1}$ and $nH$ the $n$-fold Whitney sum of $H$ with itself. In [2], it is shown that a necessary condition for $nH$ to be stably fiber homotopic trivial is that $n$ be divisible by the “Atiyah-Todd number” $M_k$, defined as follows: for a prime $p$, the highest power of $p$ dividing $M_k$ is

$$v_p(M_k) = \max\{r + v_p(r) : 1 \leq r \leq [(k - 1)/(p - 1)]\}.$$ 

In [1], this condition is shown to be sufficient as well.

The number-theoretic derivation of $M_k$ in [2] is somewhat involved. We present a simpler, and more illuminating, derivation.

Recall $K(CP^{k-1}) = \mathbb{Z}[x]/x^k$ where $x = H^* - 1$. For $nH$ to be stably fiber homotopic trivial, the “cannibalistic classes” $\theta_p(nH)$ must satisfy (see [4, p. 37])

$$\theta_p(nH) = p^n u/\psi^p(u),$$

where $u$ is some element in $1 + \tilde{K}(CP^{k-1})$. But by [4, p. 34],

$$\theta_p(nH) = (\theta_p(H))^n = \left[(x + 1)^p - 1\right]/x^n$$

$$= \left(p + \frac{p}{2} x + \cdots + \frac{p}{p - 1} x^{p-2} + x^{p-1}\right)^n$$

$$= (py + x^{p-1})^n,$$

where $y$ is an element in $1 + \tilde{K}(CP^{k-1})$. Since the elements $y^n - x^{(p-1)r} (1 \leq r \leq [(k - 1)/(p - 1)])$ form a partial basis of $\tilde{K}(CP^{k-1})$, we conclude

Received by the editors October 27, 1971.

AMS 1970 subject classifications. Primary 55F50; Secondary 55F40.

Key words and phrases. Complex projective spaces, stably fiber homotopic trivial bundles, “Atiyah-Todd” number, cannibalistic classes.

1 Supported by the National Research Council of Canada (Grant No. 67-7562). 

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that

\[ p^r \left| \binom{n}{r} \right| \text{ for all } 1 \leq r \leq \left\lfloor \frac{k-1}{p-1} \right\rfloor. \]

The binomial identity

\[ \binom{m+n}{r} = \sum_{i+j=r} \binom{m}{i} \binom{n}{j} \]

shows that, for \( p \) a fixed prime, all the \( n \)'s satisfying (*) form an ideal.

Direct inspection from the formula

\[ \binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r(r-1)\ldots1} = \frac{n}{r} \prod_{j=1}^{r-1} \frac{n-j}{j} \]

shows that \( p^i \) belongs to this ideal iff \( \lambda \geq v_p(M_k) \). This completes the derivation of \( M_k \).

Notice that if ordinary characteristic classes (or equivalently, Steenrod reduced \( p \)th power operations on Thom classes) were used instead of \( \theta_p \), one would obtain (cf. [3, p. 445]) the following weaker necessary condition:

\[(**) \quad p \left| \binom{n}{r} \right| \text{ for all } 1 \leq r \leq \left\lfloor \frac{k-1}{p-1} \right\rfloor.\]

Thus our derivation brings out the superiority of \( K \)-theory methods.

Let \( h = q^a \) be a prime power. By the same kind of argument, it can be shown that a necessary condition for \( nH^h \) to be stably fiber homotopic trivial is

\[(#) \quad v_p(n) \geq v_p(M_k) \quad \text{for any prime } p \neq q; \quad \text{and} \quad v_p(n) \geq \max\{r + v_p(r) : 1 \leq r \leq [(k-1)/(p-1)]\} \quad \text{for } p = q.\]

The method of [1] can then be used to establish that this is a sufficient condition as well. Thus the order of \( J(H^h) \) in the group \( J(CP^{k-1}) \) is completely determined.

REFERENCES


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