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FIBER HOMOTOPIC TRIVIAL BUNDLES OVER COMPLEX PROJECTIVE SPACES

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ABSTRACT. We give a more illuminating derivation of a well-known condition (obtained by M. Atiyah and J. A. Todd) for certain vector bundles over complex projective spaces to be stably fiber homotopic trivial, together with a generalization.

Let H be the Hopf line bundle over the complex projective space CP^{k-1} and nH the n-fold Whitney sum of H with itself. In [2], it is shown that a necessary condition for nH to be stably fiber homotopic trivial is that n be divisible by the "Atiyah-Todd number" M_k, defined as follows: for a prime p, the highest power of p dividing M_k is

$$\nu_p(M_k) = \operatorname{Max}\{r + \nu_p(r) : 1 \leqq r \leqq [(k-1)/(p-1)]\}.$$

In [1], this condition is shown to be sufficient as well.

The number-theoretic derivation of M_k in [2] is somewhat involved. We present a simpler, and more illuminating, derivation.

Recall $K(CP^{k-1}) = Z[x]/x^k$ where $x = H^* - 1$. For nH to be stably fiber homotopic trivial, the "cannibalistic classes" $\theta_p(nH)$ must satisfy (see [4, p. 37])

$$\theta_p(nH) = p^n u/\psi^p(u),$$

where u is some element in $1 + \tilde{K}(CP^{k-1})$. But by [4, p. 34],

$$\begin{aligned}\theta_p(nH) &= (\theta_p(H))^n = [((x+1)^p - 1)/x]^n \\ &= \left(\binom{p}{1} + \binom{p}{2}x + \cdots + \binom{p}{p-1}x^{p-2} + x^{p-1}\right)^n \\ &= (py + x^{p-1})^n,\end{aligned}$$

where y is an element in $1 + \tilde{K}(CP^{k-1})$. Since the elements $y^{n-r}x^{(p-1)r}$ $(1 \leqq r \leqq [(k-1)/(p-1)])$ form a partial basis of $\tilde{K}(CP^{k-1})$, we conclude

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that

$$(*) \qquad p^r \,\Big|\, \binom{n}{r} \quad \text{for all } 1 \leqq r \leqq \left[\frac{k-1}{p-1}\right].$$

The binomial identity

$$\binom{m+n}{r} = \sum_{i+j=r} \binom{m}{i}\binom{n}{j}$$

shows that, for p a fixed prime, all the n's satisfying (*) form an ideal. Direct inspection from the formula

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1} = \frac{n}{r}\prod_{j=1}^{r-1}\left(\frac{n-j}{j}\right)$$

shows that p^λ belongs to this ideal iff $\lambda \geqq \nu_p(M_k)$. This completes the derivation of M_k.

Notice that if ordinary characteristic classes (or equivalently, Steenrod reduced pth power operations on Thom classes) were used instead of θ_p, one would obtain (cf. [3, p. 445]) the following weaker necessary condition:

$$(**) \qquad p \,\Big|\, \binom{n}{r} \quad \text{for all } 1 \leqq r \leqq \left[\frac{k-1}{p-1}\right].$$

Thus our derivation brings out the superiority of K-theory methods.

Let $h=q^\alpha$ be a prime power. By the same kind of argument, it can be shown that a necessary condition for nH^h to be stably fiber homotopic trivial is

$$(\#) \qquad \begin{array}{ll} \nu_p(n) \geqq \nu_p(M_k) & \text{for any prime } p \neq q; \quad \text{and} \\ \nu_p(n) \geqq \operatorname{Max}\{r + \nu_p(r) : 1 \leqq r \leqq [(k-1)/h(p-1)]\} & \text{for } p = q. \end{array}$$

The method of [1] can then be used to establish that this is a sufficient condition as well. Thus the order of $J(H^h)$ in the group $J(CP^{k-1})$ is completely determined.

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