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PERTURBATIONS OF DISSIPATIVE OPERATORS WITH RELATIVE BOUND ONE[1]

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ABSTRACT. Let A be the generator of a (C_0) contraction semigroup on a Banach space. Let B be a dissipative operator with densely defined adjoint. Assume that the inequality $\|Bx\| \leqq \|Ax\| + b\|x\|$ holds on the domain of A. Then the closure of $A+B$ generates a (C_0) contraction semigroup.

Let A be the generator of a (C_0) contraction semigroup on a Banach space X. Let B be a dissipative operator on X in the sense of Lumer and Phillips [3]. Assume that $\mathscr{D}(B) \supset \mathscr{D}(A)$. Since A is closed it follows that there are constants a, $b < \infty$ such that, for every $x \in \mathscr{D}(A)$,

$$\|Bx\| \leqq a\,\|Ax\| + b\,\|x\|. \tag{1}$$

We say that B is bounded relative to A and refer to a as a relative bound.

Gustafson [1], generalizing basic work of Rellich, Kato, and others (cf. [2]), showed that if the bound a in (1) can be taken strictly less than 1 it follows that $A+B$ is the generator of a (C_0) contraction semigroup. This is known to fail for $a > 1$. On the other hand, Wüst [4] recently showed that if A and B are symmetric operators on a Hilbert space with A selfadjoint then the validity of (1) with $a=1$ implies that $A+B$ is essentially selfadjoint, i.e., has selfadjoint closure. (Kato [2] had proved a slightly weaker result, starting from the analogue of (1) with norms replaced by their squares.)

In this note we use a simplified version of Wüst's argument to extend the result to dissipative operators in a rather general Banach space setting.

THEOREM. *Let X be a Banach space. Let A and B be as above with $\mathscr{D}(B) \supset \mathscr{D}(A)$. Assume that there is a constant $b < \infty$ such that, for all $x \in \mathscr{D}(A)$,*

$$\|Bx\| \leqq \|Ax\| + b\,\|x\|. \tag{2}$$

Suppose also that the adjoint B^* has a dense domain in X^*.

Then the closure of $A+B$ is the generator of a (C_0) semigroup.

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REMARKS. 1. Our proof can easily be modified to go through under the assumption that A^*, rather than B^*, is densely defined.

2. If the space X is reflexive then A^* is densely defined. This uses only the fact that A is closed and densely defined; the familiar "graph" argument, given by von Neumann in Hilbert space, works perfectly well in any reflexive Banach space.

Thus the hypothesis on B^* is not needed if X is reflexive.

3. The result of Wüst follows from the proof of our theorem.

PROOF OF THE THEOREM. First note that $A+B$ is dissipative and therefore closable with dissipative closure [3, Lemma 3.3]. By the Hille-Yosida-Phillips characterization of semigroup generators it is enough to show that $I-(A+B)$ has dense range.

Suppose that $y^*\in X^*$ annihilates the range of $I-(A+B)$. We can find $y\in X$ with $\|y\|=\|y^*\|$ and $\langle y^*, y\rangle \geqq \frac{1}{2}\|y^*\|^2$.

Now, by the theorem of Gustafson quoted earlier, it follows from (2) that $A+tB$ is a semigroup generator for $0\leqq t<1$. Hence for each such t there is an element $x_t\in\mathscr{D}(A+tB)=\mathscr{D}(A)$ with

$$y = (1 - A - tB)x_t. \tag{3}$$

We have $\|x_t\|\leqq\|y\|$ since $A+tB$ is dissipative.

Furthermore, by (2),

$$\begin{aligned}\|Bx_t\| &\leqq \|Ax_t\| + b\,\|x_t\| \\ &\leqq \|(A+tB)x_t\| + \|tBx_t\| + b\,\|x_t\| \\ &= \|x_t - y\| + \|tBx_t\| + b\,\|x_t\|\end{aligned}$$

whence

$$\|(1-t)Bx_t\| \leqq \|x_t - y\| + b\,\|x_t\| \leqq (1+b)\,\|x_t\| + \|y\|,$$

that is,

$$\|(1-t)Bx_t\| \leqq (2+b)\,\|y\|. \tag{4}$$

Now $\{x_t\}$ is bounded, and therefore it has a subnet $x_{t'}$ which converges in the weak * sense as $t'\to 1$ to some element $x^{**}\in X^{**}$. We shall show that $(1-t')Bx_{t'}$ converges weakly to 0.

For this, suppose $z^*\in\mathscr{D}(B^*)$. Then we have

$$\begin{aligned}\langle z^*, (1-t')Bx_{t'}\rangle &= (1-t')\langle B^*z^*, x_{t'}\rangle \\ &\to 0\cdot\langle B^*z^*, x^{**}\rangle = 0.\end{aligned} \tag{5}$$

But since $\mathscr{D}(B^*)$ is dense in X^* and uniform boundedness holds by (4), a standard approximation argument shows that (5) is valid for all $z^*\in Y^*$. In particular it holds for y^*.

But

$$\begin{aligned}\langle y^*, y\rangle &= \langle y^*, (1 - A - tB)x_t\rangle \\ &= \langle y^*, (1 - A - B)x_t + (1 - t)Bx_t\rangle = \langle y^*, (1 - t)Bx_t\rangle\end{aligned}$$

since y^* kills $(1-A-B)x_t$ by assumption.

Hence

$$\tfrac{1}{2}\,\|y^*\|^2 \leqq \langle y^*, y\rangle = \lim_{t'\to 1}\langle y^*, (1 - t')Bx_{t'}\rangle = 0.$$

This shows that $y^*=0$. Hence the range of $1-A-B$ is dense. ∎

Note added in proof. On November 24, 1971, I received from Professor N. Okazawa a preprint entitled "A perturbation theorem for linear contraction semigroups on reflective Banach spaces". His main result is the same as ours (for the case of reflexive spaces).

References

1. K. Gustafson, *A perturbation lemma*, Bull. Amer. Math. Soc. **72** (1966), 334–338. MR **32** #4555.

2. T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR **34** #3324.

3. G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math. **11** (1961), 679–698. MR **24** #A2248.

4. R. Wüst, *Generalisations of Rellich's theorem on perturbation of (essentially) self-adjoint operators*, Math. Z. **119** (1971), 276–280.

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