PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33. Number 1, May 1972

WEAK DISJOINTNESS OF TRANSFORMATION GROUPS

REUVEN PELEG[1]

ABSTRACT. Two transformation groups (t.g.) are called weakly disjoint if their product is ergodic. We characterize this relation for a certain class of t.g. and then prove that for (X, T) and (Y, T) in a certain family of t.g. (X, T) and (Y, T) are disjoint iff they have no nontrivial common factor. Finally, we generalize some disjointness relations of [2] and [4].

Weak disjointness.

DEFINITION. (X, T) and (Y, T) are weakly disjoint if $(X\times Y, T)$ is ergodic. We denote this relation by $X\dot{-}Y$. (We consider only t.g.'s on compact spaces.)

The reader is referred to [2] and [3] for concepts and notation.

LEMMA 1. *If X is minimal and Y is ergodic, then $X\perp Y$ implies $X\dot{-}Y$.*

PROOF. Let $\Delta\subseteq X\times Y$ be a subflow with a nonempty interior. The minimality of X and the ergodicity of Y imply that the projections from Δ to X and Y are onto. Thus Δ must be equal to $X\times Y$ so that $X\dot{-}Y$.

The next lemma is merely a reformulation of definitions.

LEMMA 2. *X is weakly mixing iff $X\dot{-}Y$.*

In order to characterize the relation $X\dot{-}Y$ we use a method of [1] which was used later in [3], but first we need two lemmas.

LEMMA 3. *If (X, μ), (Y, ν) are probability spaces and (Y, ν) is separable then the following conditions on a measurable subset $E\subset X\times Y$ are equivalent*:

(1) *E is a rectangle (a.e. $\mu\times\nu$) of the form $X\times B$.*

(2) *$\nu(E_x)=C$ a.e. μ and $\nu(E_x\cap E_s)=C'$ a.e. $\mu\times\mu$ (E_x denotes the section at x).*

Received by the editors April 6, 1971.

AMS 1969 *subject classifications*. Primary 5482.

Key words and phrases. Transformation groups, disjointness, eigenvalues.

[1] This work was done under the supervision of Professor H. Furstenberg as a part of the author's expected Ph.D. thesis to be submitted to the Hebrew University of Jerusalem.

PROOF. (1)$\Rightarrow$(2) is obvious.

Assume (2). Let $X_1=\{x:\nu(E_x)=C\}$, $F=\{(x, s)|\nu(E_x\cap E_s)=C'\}$. Choose a dense sequence $\{A_i\}$ in the metric space of measurable sets of Y and define, for an arbitrary $\varepsilon>0$, $\Omega_i=\{x:\nu(A_i\cap E_x)<\varepsilon\}\cap X_1$. $X_1=\bigcup_i \Omega_i$ so we may find a set Ω_{i0} with positive measure. For any $(x, s)\in(\Omega_{i0}\times\Omega_{i0})\cap F$, $\nu(A_{i0}\triangle E_x)<\varepsilon$, $\nu(A_{i0}\triangle E_s)<\varepsilon$, $\nu(E_x)=\nu(E_s)=C$, and $\nu(E_x\cap E_s)=C'$. Thus

$$2\varepsilon > \nu(E_x \cap E_s) = \nu(E_x) + \nu(E_s) - 2\nu(E_x \cap E_s) = 2(C - C').$$

Since ε was arbitrary, we have $C=C'$.

Now the set $(X_1\times X_1)\cap F$ has measure 1 and for any (x, s) in it $\nu(E_x\triangle E_s)=0$. This implies (1).

REMARK. A similar proof, using the separability of $L_2(Y, \nu)$ yields a sufficient and necessary condition for a function in $L_2(X\times Y, \mu\times\nu)$ to be in $L_2(Y, \nu)$. We do not know whether the assumption of separability is essential.

LEMMA 4. *If μ is a closed ergodic invariant measure supported by the metric t.g. (X, T), ν is an invariant measure supported by (Y, T) and $E\subset X\times Y$ is closed invariant, then $\nu(E_x)$ is constant a.e.*

PROOF. $\nu(E_x)=\int k(x, y)\, d\nu(y)$ where k is the characteristic function of E. $k(x, y)$ is upper semicontinuous so $\nu(E_x)$ has the same property. Thus, for every α, $A_\alpha=\{x:\nu(E_x)\geqq\alpha\}$ is closed. A_α is a closed invariant set, so its measure is 0 or 1. This implies that $\nu(E_x)$ is constant a.e.

We turn now to settle the abelian case. In the next theorem X may be any separable metric Baire space.

THEOREM 5. *If T is abelian, (X, T) and (Y, T) are t.g. which support closed ergodic invariant measures μ, ν respectively, then $X\dot{-}Y$ iff the only common eigenvalue of (X, T) and (Y, T) is 1.*

(See [3] for the definitions of eigenvalue and eigenfunction.)

PROOF. If (X, T) and (Y, T) have a common eigenvalue $\lambda\neq 1$ with the eigenfunctions f and g respectively, then $f\bar{g}$ is a nonconstant invariant function of $X\times Y$. This is impossible when $X\dot{-}Y$ so the condition is necessary. We proceed to show sufficiency. Let $E\subseteq X\times Y$ be a subflow with a nonempty interior. We want to show $E=X\times Y$. Let $k(x, y)$ be the characteristic function of E and $K(x, s)=\int k(x, y)k(s, y)\, d\nu(y)$. By Lemmas 4 and 3 it is sufficient to prove that $K(x, s)$ is constant a.e. $\mu\times\mu$. Assume the converse and define an operator R on $L_2(x, \mu)$ by

$$Rf(x) = \int K(x, s)f(s)\, d\mu(s).$$

R is a Hilbert Schmidt operator and just as in [**1**] and [**3**, Theorem 2.5], we deduce the existence of a nonconstant function $f\in L_2(x, \mu)\cap B(X)$ which satisfies $Rf=\lambda f$ for some $\lambda\neq 0$ and $tf=\chi(t)f$ where χ is a character of T.

Define $g(y)=\int k(x,y)f(x)\,d\mu(x)$. Obviously, $g\in L_2(Y,\nu)\cap B(Y)$, $tg=\chi(t)g$ and g is not identically 0 because $\lambda f(x)=\int k(x,y)g(y)\,d\nu(y)$.

Thus, we found a common eigenvalue $\chi(t)\not\equiv 1$ so $K(x,s)$ must be constant a.e. $\mu\times\mu$.

COROLLARY 6. *If T is abelian, (X, T) and (Y, T) minimal, then $X\dot{-}Y$ iff $\gamma(X)\perp\gamma(Y)$. ($\gamma(X)$ is the maximal equicontinuous factor of X.)*

We turn now to the nonabelian case. If G is a group of unitary matrices of order $n\times n$ and $\alpha=(\alpha_1,\cdots,\alpha_n)$ is an n-tuple of complex numbers, then taking the orbit closure $\mathrm{cls}[\{\alpha A/A\in G\}]$ we get a minimal isometric transformation group.

DEFINITION. If (X, T) is a right t.g., $f_1,\cdots,f_n\in B(X)$ are independent, then $(f_1,\cdots,f_n)$ is called *n-eigenfunction* (n-e.f.) if there exists an antihomomorphism χ from T onto a group G of unitary matrices which satisfies: $t(f_1,\cdots,f_n)=(f_1,\cdots,f_n)\chi(t)$ for a comeager subset of X. χ is called an *n-eigenvalue* (n-e.v.). Recall Theorem 2.2 of [**2**]:

LEMMA 7. *(X, T) is ergodic iff 1 is a simple 1-e.v.*

LEMMA 8. *If F is an n-eigenfunction and $U\subset X$ is comeager, then there exist $x_1,\cdots,x_n\in U$ such that $\{F(x_i)\}$ is a basis of C^n.*

PROOF. Let L be the subspace spanned by $\{F(x)|x\in U\}$. If L is not the entire space one can find a vector $\alpha\neq 0$ such that the inner product $(\alpha, F(x))=0$ $(x\in U)$. This contradicts the independence of $(f_1,\cdots,f_n)=F$.

THEOREM 9. *If (X, T) and (Y, T) are t.g. supporting closed ergodic invariant measures μ, ν respectively, then $X\dot{-}Y$ iff their only common n-eigenvalue is the 1-e.v. 1.*

PROOF. The condition is necessary. Let $\chi(t)$ be a common n-e.v., $n>1$ (the case $n=1$ is simple). Let F and G be n-e.f. which correspond to $\chi(t)$. Let $\phi(x,y)=\langle F(x), G(y)\rangle$ then

$$\phi(xt,yt) = \langle F(xt), G(yt)\rangle = \langle F(x)\chi(t), G(y)\chi(t)\rangle = \langle F(y), G(y)\rangle = \phi(x,y)$$

(the author is indebted to the referee for this simple proof of the invariance). We proceed to show that this invariant function is not constant on a comeager set $A\subset X\times Y$. Assume the contrary. There exists a comeager set $V\subset Y$ such that for every $Y\in V$ the section Ay of A is comeager in X. Let $\{G(y_j)\}$, $y_1,\cdots,y_n\in V$, form a basis of C^n and choose $x_1,\cdots,x_n\in\bigcap_1^n Ay_i$ such that $\{F(x_j)\}$ is another basis of C^n. Let $0\neq\gamma\in C^n$, $\sum\gamma_j=0$ and choose $\{\delta_j\}_1^n$ to satisfy $v=\sum_1^n \gamma_j F(x_j)=\sum_1^n \delta_j G(y_j)$ then $0<\langle\langle v, v\rangle\rangle=0$ shows the contradiction.

The sufficiency is proved essentially in the same way as in Theorem 5. Let $(f_1, \cdots, f_n)$ be an orthonormal basis for the space of eigenfunctions of R which correspond to λ. There exists then a group of unitary matrices $\chi(t)$ such that $t(f_1, \cdots, f_n)=(f_1, \cdots, f_n)\chi(t)$. We define $(g_1, \cdots, g_n)$ as in Theorem 5 and find that χ is a common n-eigenvalue.

LEMMA 10. *If (X, T) is minimal, then every n-e.v. $\chi(t)$ defines an equicontinuous factor of (X, T).*

PROOF. If $F=(f_1, \cdots, f_n)$ is an n-e.f., then F must be continuous (as in [**1**, p. 506]). Thus $x \to F(x)$ defines a homomorphism between (X, T) and $(F(X), \chi(T))$.

Now we have the analogue of Corollary 6.

THEOREM 11. *If (X, T) and (Y, T) are minimal t.g. supporting invariant measures μ and v, then $X \dot{-} Y$ iff $\gamma(X) \perp \gamma(Y)$.*

Disjointness relations. If X is a minimal flow we denote by $\mathrm{PD}(X)$ all the minimal metric flows built from X by successive proximal and distal extensions and inverse limits. (In a proximal extension every two points in the same fiber are proximal; in a distal extension no two such points are proximal.)

The following theorem generalizes an unpublished result of Ellis.

THEOREM 12. *Let T be abelian, X, Y minimal with $X \perp Y$. If $X' \in \mathrm{PD}(X)$ and $Y' \in \mathrm{PD}(Y)$, then $X' \perp Y'$ iff they have no nontrivial common factor.*

PROOF. It follows from [**5**] that proximal extensions preserve disjointness in the abelian case, so we have to take care only for the distal extensions. But if $X_\alpha \perp Y_{\alpha'}$, and $X_{\alpha+1}$ is a distal extension of X_α which has no common factor with $Y_{\alpha'}$, then by Corollary 6, $X_{\alpha+1} \perp Y_{\alpha'}$. $X_{\alpha+1} \times Y_{\alpha'}$ as a distal extension of $X_\alpha \times Y_{\alpha'}$ which is minimal is semisimple too. So it must be minimal and $X_{\alpha+1} \perp Y_{\alpha'}$. The proof is completed by induction.

LEMMA 13. *Let X and Y be minimal t.g. If X is distal and it is disjoint from the maximal distal factor of Y then $X \perp Y$.*

PROOF. Let D be the universal distal t.g., z the distal part of Y and $\phi: D \to X \times Z$ a homomorphism. Now we have the following diagram:

$$\begin{array}{ccccc} D & \xrightarrow{\phi} & X \times Z & \xrightarrow{\pi} & Z \\ & & & \nearrow_{\alpha} & \\ & & Y & & \end{array}$$

(π is the projection.) By [7], $\{(d, y)/\pi\phi(d)=\alpha(y)\}$ is minimal. Therefore, $\{(x, z, y)/z=\alpha(y)\}$ is minimal and so $X\times Y$ is minimal too.

THEOREM 14. *Let X and Y be metric minimal t.g., X distal. Then $X\perp Y$ iff $\gamma(X)\perp\gamma(Y)$.*

PROOF. Distal minimal t.g. always support an invariant measure ([1], [5]) so if Z is the distal part of Y then, by Theorem 11, $X\perp Z$ iff $\gamma X\perp\gamma Z$ and the preceding lemma finishes the proof.

Next, we want to get some disjointness relations for a general T. If A is any family of t.g. with phase group T, then A_I will denote those t.g. of A which support an ergodic measure. $\mathcal{M}$, $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{W}$ denote respectively, the minimal, distal, ergodic, and weakly mixing t.g.

LEMMA 15. *If X, $Y\in\mathcal{E}_I$ and $X\dot{-}Y$, then $X\times Y\in\mathcal{E}_I$.*

PROOF. The proof of Theorem 9 shows that $\mu\times\nu$ is a closed ergodic invariant measure supported by $X\times Y$. The lemma follows now by Proposition 2.6 of [3].

The next theorem generalizes the corresponding results of [2] and [4]:

THEOREM 16. (1) $(\mathcal{M}^{\dot{-}})_I=\mathcal{W}_I$.
(2) $\mathcal{W}_I\times\mathcal{W}_I=\mathcal{W}_I$.
(3) $X\in\mathcal{W}_I\Rightarrow X^n\in\mathcal{W}_I$.
(4) $\mathcal{W}_I\dot{-}\mathcal{E}_I\cup\mathcal{M}$.
(5) $\mathcal{W}_I\perp\mathcal{D}\cap\mathcal{M}$.
(6) $(\mathcal{D}^{\dot{-}})_I=\mathcal{W}\cap\mathcal{M}_I$.

PROOF. (1) and (2) follow by Lemma 15 and Theorem 9. (3) is a special case of (2).

(3) implies that if $X\in\mathcal{W}_I$ then $N(A, B)$ contains a left translation of every finite subset of T. This implies $\mathcal{W}_I\dot{-}\mathcal{M}$. $\mathcal{W}_I\dot{-}\mathcal{E}_I$ follows from Theorem 9 and thus we established (4).

For (5) use [4] and Furstenberg's structure theorem as in [2, Theorem 3].

(6) is a consequence of (4) and (5) (see Theorem 4.4 in [4]).

REMARK. The subscript I is essential. If we take X to be the unit circle and T is generated by the two homeomorphisms

$$\exp 2\Pi ix \rightarrow \exp 2\Pi i(x+\Pi) \quad \text{and} \quad \exp 2\Pi ix \rightarrow \exp 2\Pi ix^2,$$

then $(X, T)\in\mathcal{W}$ but $X\times X\times X\notin\mathcal{E}$.

Our last theorem is independent from the rest of the paper and gives some more information about $(\mathcal{W}\cap\mathcal{M})^{\perp}$. Call (X, T) weakly distal if there exists an $x_0\in X$ with card $x_0P(X)\leqq\aleph_0$. ($P(X)$ is the proximal relation of X.)

THEOREM 17. *If (X, T) is a weakly distal minimal metric abelian t.g. then $X \perp \mathscr{W}' \cap \mathscr{M}$.*

PROOF. Let $Y \in \mathscr{W}' \cap \mathscr{M}$ and let Δ be a minimal subflow of $X \times Y$. We have to show $\Delta = X \times Y$. Choose $(x_0, y_0) \in \Delta$ with card $x_0 P(X) \leqq \aleph_0$ and denote by π the projection from Δ onto Y. By [**6**], $y_0 P(Y) = A$ is a residual set and, since $\pi[(x_0, y_0)P(\Delta)] = A$, we can find for every $y \in A$ an $x_i \in x_0 P(X)$ so that (x_0, y_0) and (x_i, y) are proximal in Δ. Define

$$B_i = \text{cls}\{y : ((x_0, y_0), (x_i, y)) \in P(\Delta)\}.$$

Obviously, $\{x_i\} \times B_i \subset \Delta$ and $\bigcup B_i \supset A$ so that we may find a B_j with a non-empty interior. Let $U \times V$ be an open rectangle in $X \times Y$. $N(x_j, U)$ is discretely syndetic and $N(B_j, V)$ is discretely replete so that $N(x_j, U) \cap N(B_j, V) \neq \varnothing$. Thus, the orbit of $\{x_j\} \times B_j$ is dense and $\Delta = X \times Y$.

Appendix. Using the remark after Lemma 3 and the technique of Theorem 5, one can show that if (X, T, μ) and (Y, T, ν) are ergodic *processes*, then every eigenvalue of $(X \times Y, T, \mu \times \nu)$ is the product of an e.v. of X with an e.v. of Y. Furthermore, if $\mu \times \nu$ is ergodic, then every e.f. of $X \times Y$ is the product of an e.f. of X with an e.f. of Y.

REFERENCES

1. H. Furstenberg, *The structure of distal flows*, Amer. J. Math. **85** (1963), 477–515. MR **28** #602.

2. ———, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49. MR **35** #4369.

3. H. B. Keynes and J. B. Robertson, *Eigenvalue theorems in topological transformation groups*, Trans. Amer. Math. Soc. **139** (1969), 359–369. MR **38** #6029.

4. H. B. Keynes, *The structure of weakly mixing transformation groups* (preprint).

5. R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

6. R. Ellis and H. Keynes, *A Characterization of the equicontinuous structure relation* (preprint).

7. L. Shapiro, *Distal and proximal extensions of minimal flows*, Math. Systems Theory **5** (1971), 76–88.

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL