A NOTE ON THE ASYMPTOTIC BEHAVIOR
OF AN INTEGRAL EQUATION

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Abstract. Assume the existence and boundedness of a solution
to an integral equation. Conditions are found which ensure the
solution has a limit at infinity.

1. Introduction. Consider the integral equation

\[ x'(t) + \int_{-\infty}^{\infty} g(x(t - \xi)) \, dA(\xi) = f(t) \]

where \( g, A \) and \( f \) are prescribed real functions, and \( x \) is a bounded solution
of (1.1) on \((-\infty, \infty)\). We use NBV to denote normalized bounded
variation, LAC for absolutely continuous on any compact interval of
\((-\infty, \infty)\), \( V(A, [t_1, t_2]) \), the total variation of \( A \) on \([t_1, t_2]\), and \( V(A) = V(A, (-\infty, \infty)) \). Consider the following hypotheses.

\( H(f) : f \in L^\infty(-\infty, \infty), \lim_{t \to \infty} f(t) = f(\infty) \) exists.

\( H(g) : g \in C(-\infty, \infty), S = \{ c | g(c)A(\infty) = f(\infty) \} \) is nonempty and contains no interval.

\( H'(g) : g \in C(-\infty, \infty), S \) contains exactly one point.

\( H(A) : A(t) = A_1(t) + A_2(t), A_1(t) = 0 (-\infty < t \leq 0), \]

\[ A_1(t) = \rho_1 > 0 (0 < t < \infty), \quad A_2(t) \in NBV(-\infty, \infty), \]

and \( V(A_2) = \rho_2 < \rho_1 \).

Theorem 1. Let \( H(f), H(g) \) and \( H(A) \) hold. Let \( x(t) \in LAC(-\infty, \infty) \cap \)
\( L^\infty(-\infty, \infty), \) let \( x'(t) \) exist for each \( t \) and let \( x(t) \) satisfy (1.1) on \((-\infty, \infty)\).

Then

\[ \lim_{t \to \infty} x(t) = c, \quad \lim_{t \to \infty} x'(t) = 0 \]

for some \( c \in S \).

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Theorem 2. Let $H(f)$, $H'(g)$ and $H(A)$ hold. Let $x(t) \in \text{LAC}(-\infty, \infty) \cap L^\infty(-\infty, \infty)$ satisfy (1.1) a.e. on $(-\infty, \infty)$. Then

$$\lim_{t \to \infty} x(t) = c, \quad \lim_{t \to \infty} \left[ \text{ess sup}_{t \leq s \leq t} |x'(s)| \right] = 0$$

for $c \in S$.

Theorem 1 was obtained by Levin and Shea [2] under the additional hypothesis that $g$ is either strictly increasing or strictly decreasing. Theorem 2 assumes that $x$ is a solution of (1.1) only a.e. on $(-\infty, \infty)$ and was obtained independently by Levin and Shea and by the author. A proof appears in [1].

It is shown in [1] that the requirement $\rho_2 < \rho_1$ is necessary in $H(A)$. It is obvious that the requirement, $x$ a bounded solution, is necessary in Theorems 1 and 2. For if $g(x) = -x$, $A(t) = 0$ ($-\infty < t \leq 0$), $A(t) = 1$ ($0 < t < \infty$), $f(t) = 0$, then $c = 0$ and (1.1) reduces to $x'(t) - x(t) = 0$ which has $x(t) = e^t$ as a solution.

For further discussion of (1.1) and references to applications in ordinary differential equations and Volterra equations see [1] and [2].

2. Proof of Theorem 1. We first note that $g(c)$ is the same for all $c$ in $S$. Let $x(t) \in \text{LAC}(-\infty, \infty) \cap L^\infty(-\infty, \infty)$ satisfy (1.1). Let

$$\alpha = \sup_{-\infty < t < \infty} |g(x(t)) - g(c)|$$

where $c \in S$, $\bar{z} = \lim_{t \to -\infty} |g(x(t)) - g(c)|$. Suppose $\bar{z} > 0$. Let $\varepsilon > 0$, and, using $\rho_2 < \rho_1$, let $T_\varepsilon$ be chosen such that

$$(\rho_2 - \rho_1)\bar{z} + (\rho_2 + \rho_1 + \alpha + 1)\varepsilon < -d$$

for some positive constant $d$,

$$\sup_{t > T_\varepsilon} |g(x(t)) - g(c)| \leq \bar{z} + \varepsilon,$$

(2.1)

$$V(A_2, [T, \infty)) < \varepsilon, \quad \sup_{t > T_\varepsilon} |f(t) - f(\infty)| < \varepsilon.$$ By definition of $\bar{z}$, there exist $\{t_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n\}_{n=1}^{\infty}$ with $t_1 > 2T_\varepsilon$, $\varepsilon_1 < \varepsilon$ such that $t_n \to \infty$ ($n \to \infty$), $\varepsilon_n \to 0$ ($n \to \infty$) and either

$$(2.2) \quad \bar{z} - \varepsilon_n < g(x(t_n)) - g(c) < \bar{z} + \varepsilon_n, \quad \text{or}$$

$$(2.3) \quad -\bar{z} - \varepsilon_n < g(x(t_n)) - g(c) < -\bar{z} + \varepsilon_n \quad (n = 1, 2, \cdots).$$

Suppose (2.2) holds. From (1.1) and $H(A)$ we have

$$x'(t) + \rho_1 g(x(t)) = -\int_{-\infty}^{x(t)} g(x(t - \xi)) \, dA_2(\xi) + f(t).$$
and using \( f(\infty) = g(c)A(\infty) = g(c)(\rho_1 + A_2(\infty)) \).

\[
x'(t) + \rho_1(g(x(t)) - g(c)) = -\int_{-\infty}^{\infty} (g(x(t - \xi)) - g(c)) dA_2(\xi) + f(t) - f(\infty).
\]

Let \( t = t_n \). By (2.1) and (2.2) we have

\[
x'(t_n) + \rho_1(\tilde{\alpha} - \varepsilon_n) \\
\leq -\int_{-\infty}^{T_\varepsilon} (g(x(t_n - \xi)) - g(c)) dA_2(\xi) + f(t_n) - f(\infty) \\
\leq (\tilde{\alpha} + \varepsilon)\rho_2 + \alpha\varepsilon + \varepsilon,
\]

and hence by (2.1),

\[
x'(t_n) < -d.
\]

By a similar argument, \( t > t_n \) and \( x(t) = x(t_n) \) imply \( x'(t) < -d \). Hence \( t > t_n \) and \( x \) continuous imply \( x(t) < x(t_n) \). In particular,

\[
-\|x\|_\infty \leq x(t_{n+1}) < x(t_n).
\]

Thus there exists \( x^* \) such that \( x(t_n) \downarrow x^\ast \). By (2.2), \( g(x^*) - g(c) = \tilde{\alpha} \). Since \( g \in C(-\infty, \infty) \), there exists \( x^{**} > x^\ast \) such that, for \( x^\ast < x(t) < x^{**} \), \( \tilde{\alpha} - \varepsilon < g(x(t)) - g(c) < \tilde{\alpha} + \varepsilon \). Choose \( N \) such that \( x^\ast < x(t_N) < x^{**} \) and hence for \( t > t_N \), \( x^\ast < x(t) < x^{**} \). Then as in the proof of (2.4), we have \( t > t_N \) implies \( x'(t) < -d \). This contradicts \( x \in L^\infty \), hence (2.2) is impossible.

Similarly (2.3) is impossible, \( \tilde{\alpha} = 0 \) and \( \lim_{t \to \infty} g(x(t)) = g(c) \), where \( g(c) = g^* \) is independent of \( c \in S \). Then \( \lim_{t \to \infty} g(x(t)) = g^* \) implies \( x(t) \) approaches a value of \( x \) such that \( g(x) = g^* \). That is, \( \lim_{t \to \infty} x(t) = c \) for some \( c \in S \). From (1.1) we then have \( \lim_{t \to \infty} x'(t) = 0 \).

BIBLIOGRAPHY


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