COHOMOLOGY OPERATIONS ON p-FOLD SUMS

EMERY THOMAS

Abstract. A formula is given for evaluating higher order cohomology operations on integral classes that are p-fold multiples, p a prime.

Let $\Omega$ be an $n$th order ($n \geq 2$) cohomology operation defined on integral cohomology classes, given by a relation

$$
\sum_i \alpha_i \Phi_i = 0,
$$

where each $\Phi_i$ is an $(n-1)$st order (integral) operation and each $\alpha_i$ is an element of the mod $p$ Steenrod algebra $A$. We will think of each operation $\Phi_i$ as being defined on integral classes of some fixed degree, $q$. We suppose that $\Phi_i$ is the suspension of an operation $\Psi_i$, defined on classes of degree $q+1$, and that relation (1) desuspends. In particular, each $\Phi_i$ is additive. Suppose now that $X$ is a space and $u \in H^q(X, Z)$ is a class such that $\Phi_i(u)$ is defined for each $i$. Of course $\Omega$ is not necessarily defined on $u$, but $\Omega$ is defined on $pu$, since each $\Phi_i$ is additive. Our problem is: how does one compute $\Omega(pu)$?

In [2] a morphism $e: A \to \mathcal{A}$ is defined, of degree $-1$, characterized by the following properties:

(i) $e$ is a derivation of the graded algebra $A$.

(ii) If $p=2$, $e(Sq^n) = Sq^{n-1}$, $n \geq 1$; if $p>2$, $e(\beta_p^i) = 1$, $e(P^i) = 0$, $i \geq 0$, where $\beta_p$ denotes the mod $p$ Bockstein. (For $p=2$, $e$ is the morphism $\hat{e}$ considered by Kristensen [1].)

We now can state our result.

Theorem. Let $\Omega$ be an operation associated with relation (1) and let $u \in H^q(X; Z)$ be a class in the domain of each $\Phi_i$. Then,

$$
(-1)^s \sum e(\alpha_i) \Phi_i(u) \subseteq \Omega(pu),
$$

where $s = q + \deg \Omega$.
Proof. Let $B$ denote a universal example for the operations $\{\psi_i\}$ defined on integral classes of degree $q+1$. Thus, $B$ is an $(n-1)$st stage Postnikov system over $K(Z, q+1)$ (note [3]). By hypothesis, there are classes $\psi_i \in H^*(B; Z_p)$ such that

$$\sum (-1)^a \psi_i = 0, \quad \text{where } a_i = \deg \psi_i.$$

Suppose $d_i = \deg \psi_i$, and set $C = \times_i K(Z_p, q+1 + d_i)$. If we let $\psi = \{\psi_i\}$, we then have a map $\psi : B \to C$. Denote by $\psi^* : H^*(B; Z_p) \to H^*(C; Z_p)$ the pullback of cohomology classes.

The principal fibration with $\psi$ as classifying map. By hypothesis, there is a class $\lambda \in H^{s+1}(E; Z_p)$, thought of as a map

$$E \xrightarrow{\lambda} K(Z_p, s + 1) = K,$$

such that $i^* \lambda = \gamma = \{a_i\} \in H^{s+1}(\Omega C; Z_p)$.

Now take the loops of (2); we obtain a fibration

$$\Omega C \xrightarrow{i'} \Omega E \xrightarrow{\pi'} \Omega B,$$

and if we denote by $\sigma$ the loop homomorphism in cohomology, then $\sigma \psi = \phi = (\psi_i)$, and $\sigma \lambda = \omega$, a representative for $\Omega$.

Consider the following commutative diagram:

$$\begin{array}{ccc}
\Omega E & \xrightarrow{\pi'} & \Omega B \\
\downarrow \omega & & \downarrow t \\
\Omega K & \xrightarrow{k} & \Omega L \\
\downarrow \lambda & & \downarrow \\
\Omega \omega & \xrightarrow{j} & \Omega \omega \\
\end{array}$$

Here the lower line is the principal fibration sequence for the map $\omega$. ($\Omega L$ is a loop space since $\omega$ is stable.) Since the right-hand square commutes, the map $t$ exists. Now apply the functor $[X, \ ]$ to (3); we obtain the following commutative diagram with each row an exact sequence:

$$\begin{array}{ccc}
[X, \Omega E] & \xrightarrow{\pi'_*} & [X, \Omega B] \\
\downarrow \omega_* & & \downarrow t_* \\
H^0(X; Z_p) & \xrightarrow{k_*} & [X, \Omega L] \\
\downarrow \alpha_* & & \downarrow j_* \\
H^*(X; Z_p) & \xrightarrow{\alpha_*} & [X, \Omega C] \\
\end{array}$$

By hypothesis, there is a class $v \in [X, \Omega B]$ such that $v$ goes to $u$ in $H^0(X; Z)$. Since $p[X, \Omega C] = 0$, by exactness there is a class $x \in [X, \Omega E]$ such that $\pi'_*(x) = pv$. And by definition $\omega_*(x) \in \Omega(pu)$. 

On the other hand, consider \( y = t^* \in [X, \Omega L] \). We have \( j^*(y) = \phi_*(v) = \{ \phi_i(v) \} \) where \( \phi_{t^*}(v) \in \Theta_i(u) \). Also, \( k_* \omega(x) = py \). Thus, by Corollary 3.7 of [2],

\[
\omega(x) = (-1)^t \sum_i \epsilon(x_i) \phi_i(v),
\]

which completes the proof.

**Example.** The simplest example of interest is the relation

\[
(4) \quad Sq^2 Sq^2 = 0, \quad \text{on integral classes.}
\]

Since \( \epsilon(Sq^2) = Sq^1 \), we then have the result:

\[
Sq^a H^*(X; \mathbb{Z}) \subset \Omega(2H^*(X; \mathbb{Z})),
\]

where \( \Omega \) is the secondary operation given by (4).

**Remark.** As pointed out by F. Peterson, if (1) is in fact a relation that holds on mod \( p \) classes, then \( \Omega(pu) = 0 \). For let \( \rho \) denote the cohomology homomorphism induced by the coefficient group epimorphism \( \mathbb{Z} \to \mathbb{Z}_p \). Then for any integral class \( v \) in the domain of \( \Omega \), \( \Omega(v) = \Omega(\rho v) \), and hence

\[
\Omega(pu) = \Omega(\rho(pu)) = \Omega(0) \equiv 0.
\]

**References**


**Department of Mathematics, University of California, Berkeley, California 94720**