NOTE ON THE PROJECTIVE LIMIT
ON SMALL CATEGORIES
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Abstract. Let $X$ be a small category and let $\text{Ab}^X$ be the category of covariant functors on $X$ with values in $\text{Ab}$. Consider the projective limit functor $\text{proj} \lim_x : \text{Ab}^X \to \text{Ab}$. The categories $X$ for which $\text{proj} \lim_x$ is exact are characterized, proving a conjecture of Oberst.

In [Bull. Amer. Math. Soc. 74 (1968), 1129–1132], U. Oberst formulated a conjecture on the exactness of the projective limit functor on the category of functors on a small category with values in the category of abelian groups.

In this note we give a proof of his conjecture. Some of the lemmas seem to have been proved by U. Oberst and J. R. Isbell by other methods.

Theorem. If $X$ is a small connected category and $\text{Ab}$ is the category of abelian groups, then the two following conditions are equivalent:

(i) For all $F \in \text{ob} \text{Ab}^X$, $\text{proj} \lim^{(i)} F = 0$, for all $i \geq 1$.

(ii) There exists $y \in \text{ob} X$ such that

1. for all $x$ there exists $\xi \in X(y, x)$;

2. every diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & X' \\
\downarrow & & \downarrow \\
y & \xleftarrow{u} & x'
\end{array}
\]

in $X$ can be completed to a commutative diagram

\[
\begin{array}{ccc}
y & \xleftarrow{u} & x' \\
\downarrow & & \downarrow \\
x & \xrightarrow{u} & x'
\end{array}
\]

in $X$;

3. there exists $e \in X(y, y)$ such that, for all $\xi \in X(y, y)$, $\xi e = e$.

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Proof. Since (ii) implies that

$$\text{proj lim } F = H^0(X(y, y), F(y)) = \{x \in F(y) \mid F(x)(x) = x\}$$

$$= \{F(x)(\beta) \mid \beta \in F(y)\},$$

it is trivial to see that (ii) => (i).

To prove that (i) implies (ii), let $F$ be the object of $\text{Ab}^X$ defined by

$$F(x) = \bigsqcup_{\xi \in \text{ob}(X(x))} Z \xi$$

with $Z \xi = Z$ for all $\xi \in \text{ob}(X(x)).$

Consider the obvious epimorphism $\rho: F \to Z$ with $Z$ the constant object of $\text{Ab}^X$. Since proj lim is exact we have that $\rho^*: \text{proj lim } F \to \text{proj lim } Z = Z$ is an epimorphism. Therefore there exists $x \in \text{proj lim } F$ with $\rho^*(x) = 1$.

If $\pi_x: \text{proj lim } F \to F(x)$ is the canonical homomorphism, then for all $x \in X$, $\pi_x = \pi_x(x) \in F(x)$ is nonzero. Now

$$\alpha_x = \sum_{i=1}^n \sum_{j=1}^m \alpha_x(y_i) \xi_{ij},$$

where $\xi_{ij} \in X(y_i, x)$, $\alpha_x(y_i) \in Z$, and $\sum \alpha_x(y_i) = 1$.

For at least one $i$ we must have $\sum_{j=1}^m \alpha_x(y_i) \neq 0$ and we may assume that $\alpha_x(y_i) \neq 0$ for $1 \leq j \leq m' \leq m$.

If the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\psi} & X' \\
\downarrow{\phi} & & \downarrow{\psi} \\
U & \xrightarrow{\rho} & U
\end{array}$$

is in $X$, then $\phi^* \alpha_x = \alpha_u = \psi^* \alpha_x$, and therefore

$$\sum_{j} \alpha_j = \sum_{j} \alpha_j = \sum_{j} \alpha_j = \alpha_x(y_i).$$

Since $X$ is connected it follows that $\alpha_x(y_i) \neq 0$ for all $x \in X$ and there exist $\xi_{ij} \in X(y_i, u)$ with the corresponding $\alpha_x(y_i) \neq 0$. Consequently there exist $\xi_{ij} \in X(y_i, x)$, $\xi_{ij} \in X(y_i, x')$ with $\phi \xi_{ij} = \psi \xi_{ij} = \psi \xi_{ij}$, i.e. the above diagram can be completed to

$$\begin{array}{ccc}
X & \xrightarrow{\phi} & X \\
\downarrow{\psi} & & \downarrow{\psi} \\
U & \xrightarrow{\rho} & U
\end{array}$$

We have proved (ii)(2) and at the same time (ii)(1). We need only prove
(ii)(3). Let $F_1$ be the object of $\text{Ab}^X$ defined by

$$F_1(x) = \bigoplus_{\xi \in \mathcal{X}(y, x)} Z$$

with $y = y_i$ (i.e. the $y_i$ picked above).

By (ii)(1) there exists an epimorphism $F_1 \to Z$ in $\text{Ab}^X$. Since, by assumption, $Z$ is projective as an object of $\text{Ab}^X$, $Z$ is a direct summand of $F_1$, therefore $Z$ is a direct summand of $F_1(y)$, as an $\mathcal{X}(y, y)$-module. But $F_1(y)$ can be identified with the monoid algebra $Z[\mathcal{X}(y, y)]$ and it therefore follows that the cohomology of the monoid $M = \mathcal{X}(y, y)$ is trivial.

**Lemma A.** If a monoid $M$ is cohomologically trivial, then there exists an $e \in M$ such that, for all $\xi \in M$, $\xi e = \xi$.

**Proof.** Consider the epimorphism $Z[M] \to Z$. Since cohomology is trivial, the corresponding homomorphism $H^0(M, Z(M)) \to H^0(M, Z)$ is an epimorphism. Now $H^0(M, Z(M)) = \{\sum_{i=1}^{n} \alpha_i \xi_i \mid \xi_i \in Z, \xi_i \in M\}$ such that, for all $\xi \in M$ and all $i$, $\xi \xi_i = \xi_j$ for some $j = j_i$ with $\alpha_i = \alpha_{j_i}$.

It follows that there exists an element $\sum_{i=1}^{n} \alpha_i \xi_i \in Z(M)$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that for all $\xi \in M$, $\xi \xi_i = \xi_{\sigma_i(i)}$ where $\sigma_i$ is a permutation of \{1, 2, \ldots, n\}.

Let $S(n)$ be the symmetric group, then the correspondence $\xi \to \sigma_i$ gives a homomorphism $\sigma : M \to S(n)$ since $\xi^i \xi \xi_i = \xi^i \xi_{\sigma_i(i)} = \xi_{\sigma_i^i(i)}$.

Let $H = \text{im} \sigma$ then $H$ is a subgroup of $S(n)$. This follows from the fact that by the theorem of Lagrange, any submonoid of a finite group is a subgroup.

**Sublemma B.** If $M \to H$ is a surjective homomorphism of monoids and if $M$ is cohomologically trivial, then $H$ is cohomologically trivial.

**Proof.** This follows from $H^0(M, -) = H^0(H, -)$ in the category of $H$-modules. \(\text{Q.E.D.}\)

**Sublemma C.** If a group $G$ is cohomologically trivial, then $G = \{1\}$.

**Proof.** As above there exists an element $\sum_{i=1}^{n} \alpha_i \xi_i \in Z(G)$ such that for all $\xi \in G$, $\xi \xi_i = \xi_{\sigma_i(i)}$ and $\alpha_i = \sigma_{\alpha_i(i)}$. Since for all $i$, $j$ there exists an element $\xi$ with $\sigma_i(i) = j$, we have $\alpha_i = \alpha_j$ for all $i$, $j$ and $G = \{\xi_1, \ldots, \xi_n\}$. Since

$$1 = n \cdot \alpha_1 = |G| \cdot \alpha_1$$

it follows that $|G| = 1$ and $\alpha_1 = 1$, and consequently $G = \{1\}$. \(\text{Q.E.D.}\)

Combining B and C we find that $\sigma_i = 1$ for all $i \in M$. This of course means that for all $\xi \in M$, $\xi \xi_i = \xi_i$. Put $\varepsilon = \xi_i^i$ for some $i$, and we have proved A. \(\text{Q.E.D.}\)

This ends the proof of the Theorem since A implies (ii)(3). \(\text{Q.E.D.}\)

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