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NORMALLY CLOSED SATURATED FORMATIONS

MARK P. HALE, JR.

ABSTRACT. Theorems of Gaschütz and Baer concerning the Frattini subgroup, the hypercenter, and nilpotent and supersolvable subgroups are extended to normally closed saturated formations.

The hypercenter, derived group, and Frattini subgroup are all related to nilpotency of a finite group. In [4], Gaschütz studied their relations, and found a sufficient condition for a normal subgroup of a group to be nilpotent. By replacing the notion of centrality with $\mathfrak{F}$-centrality, we give theorems which extend the results of Gaschütz to normally closed saturated formations containing all nilpotent groups, and strengthen his nilpotency criterion. A theorem of Baer, on supersolvably immersed subgroups, is also extended to saturated formations.

In this paper, all groups are finite. A *class* of groups is understood to include, with G, all groups isomorphic to G. If M is a subgroup of G, core(M) is the intersection of the G-conjugates to M, and is the largest normal subgroup of G contained in M. A *chief factor* of G is a quotient H/K, where $K \trianglelefteq G$ and H/K is minimal normal in G/K. If H/K is a chief factor with $L \leqq K$ and $H \leqq M$, we say H/K lies above L and below M. A subgroup $M < G$ is a *supplement* to H/K in G if $MH = G$ and $M \cap H \geqq K$. If $M \cap H = K$, M is a *complement* to H/K. Other notation and terminology is standard, and can be found in [6].

Although the results below concern classes of solvable groups, it is not assumed that all groups dealt with are solvable.

1. Let $\mathfrak{X}$ be a class of groups. A maximal subgroup M of G is *$\mathfrak{X}$-normal* if $G/\text{core}(M)$ is in $\mathfrak{X}$, and *$\mathfrak{X}$-abnormal* otherwise. We define $T_{\mathfrak{X}}(G)$ to be the intersection of G with all $\mathfrak{X}$-normal maximal subgroups of G. If $\mathfrak{X}$ is the class of all groups, $T_{\mathfrak{X}}(G)$ is the *Frattini subgroup*, $\Phi(G)$. Many useful properties of $\Phi(G)$ apply to $T_{\mathfrak{X}}(G)$; we collect them here.

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PROPOSITION 1. *Let $\mathfrak{X}$ be an arbitrary class of groups.*

(a) *$T_{\mathfrak{X}}(G)$ is characteristic in G.*

(b) *If $N \trianglelefteq G$, $N \leqq T_{\mathfrak{X}}(G)$, then $T_{\mathfrak{X}}(G/N) = T_{\mathfrak{X}}(G)/N$.*

(c) *If N is minimal normal in G, then $N \leqq T_{\mathfrak{X}}(G)$ if and only if N is not supplemented by an $\mathfrak{X}$-normal maximal subgroup of G.*

(d) *For any homomorphism θ of G, $T_{\mathfrak{X}}(G\theta) \geqq T_{\mathfrak{X}}(G)\theta$.*

PROOF. For automorphisms θ of G, $M\theta$ is $\mathfrak{X}$-normal if M is; (a) follows. For (b), we observe that if $N \leqq M$, M is $\mathfrak{X}$-normal in G if and only if M/N is $\mathfrak{X}$-normal in G/N. A maximal subgroup M supplements a minimal normal subgroup N if and only if it does not contain it; this proves (c). Since pre-images of $\mathfrak{X}$-normal maximal subgroups are $\mathfrak{X}$-normal, the pre-image of $T_{\mathfrak{X}}(G\theta)$ contains $T_{\mathfrak{X}}(G)$, proving (d) and the proposition.

The third part of the lemma above yields the familiar fact that a chief factor M/N has a supplement in G/N if and only if M/N is not a *Frattini-factor*, i.e. is not contained in $\Phi(G/N)$. We will need the next lemma in later sections.

LEMMA 2. *If the chief factor H/K is supplemented by a maximal subgroup M of G and $G/\mathrm{core}(M)$ is solvable, then H/K is Abelian, and M complements H/K.*

PROOF. We may assume that $K=1$ and H is minimal normal in G. Let $L=\mathrm{core}(M)$. Since G/L is solvable and $HL/L \neq 1$, we have $H' < H$ and so $H'=1$. Now $H \cap M$ is normal in M and centralized by H, so $H \cap M$ is normal in $G=HM$; thus $H \cap M = 1$.

2. A nonempty class $\mathfrak{F}$ of solvable groups is a *formation* if:

(1) $\mathfrak{F}$ contains all homomorphic images of groups in $\mathfrak{F}$, and

(2) if G/M and G/N are in $\mathfrak{F}$, $G/(M \cap N)$ is in $\mathfrak{F}$.

Each group G has a smallest normal subgroup $G_{\mathfrak{F}}$ such that $G/G_{\mathfrak{F}}$ is in $\mathfrak{F}$. If $\mathfrak{F}$ contains all normal subgroups of groups in $\mathfrak{F}$, we say $\mathfrak{F}$ is *normally closed.* A formation is *saturated* if $G/\Phi(G)$ in $\mathfrak{F}$ implies G belongs to $\mathfrak{F}$. All saturated formations can be constructed by the following recipe, first given in [**5**]. For each prime p, a formation $f(p)$ is specified. The class $\mathfrak{F}(f)$ of all solvable groups G with $G/O_{p',p}(G) \in f(p)$ for each p is a saturated formation, locally defined by $\{f(p)\}$. If it is required that $f(p) \subseteq \mathfrak{F}(f)$, and that $f(p)$ contain each group G for which $G/O_p(G) \in f(p)$, then there is a unique choice for the formations $\{f(p)\}$. When we speak below of a saturated formation, it is this choice of the formations $\{f(p)\}$ to which we refer. This background material is to be found in [**2**] and [**8**], or [**3**].

A chief factor H/K of p-power order is $\mathfrak{F}$-*central* if $\mathrm{Aut}_G(H/K) = G/C_G(H/K)$ lies in $f(p)$. A group G belongs to a saturated formation $\mathfrak{F}$ if and only if each supplemented factor of G is $\mathfrak{F}$-central. The next lemma

shows that our definition of $\mathfrak{F}$-normal coincides with that given for saturated formations in [2].

PROPOSITION 3. *Let $\mathfrak{F}$ be a saturated formation. A maximal subgroup M of G is $\mathfrak{F}$-normal if and only if M complements an $\mathfrak{F}$-central chief factor of G.*

PROOF. Let $J=\text{core}(M)$. If M is $\mathfrak{F}$-normal, and K/J is minimal normal in G/J, M complements K/J, and since $G/J\in\mathfrak{F}$, K/J is $\mathfrak{F}$-central. Conversely, suppose $MH=G$, $M\cap H=K$, H/K is minimal normal in G/K, and H/K is $\mathfrak{F}$-central. Then $L=C_G(H/K)\geqq H$, so $L=L\cap MH=(L\cap M)H$. Since $L\cap M/K$ is normal in G/K, $L\cap M$ is normal in G. Thus $L\cap M\leqq J$. On the other hand, $[H,J]\leqq H\cap M=K$, so $HJ=C_G(H/K)$. Thus $\text{Aut}_G(H/K)=G/HJ$ belongs to $f(p)\subseteq\mathfrak{F}$, and since $HJ/J\simeq H/H\cap J=H/K$, HJ/J is $\mathfrak{F}$-central, so $G/J\in\mathfrak{F}$.

For $\mathfrak{F}$ a saturated formation, we define the *$\mathfrak{F}$-Frattini subgroup* $\Phi_{\mathfrak{F}}(G)$, to be the subgroup $T_{\mathfrak{F}'}(G)$, where $\mathfrak{F}'$ is the class of all groups not in $\mathfrak{F}$. We note that $\Phi_{\mathfrak{F}}(G)\cap T_{\mathfrak{F}}(G)=\Phi(G)$; also by (c) of Proposition 1 and Lemma 2, every supplemented chief factor below $\Phi_{\mathfrak{F}}(G)$ is complemented by an $\mathfrak{F}$-normal maximal subgroup and is therefore $\mathfrak{F}$-central. We now give a strengthened test for saturation.

THEOREM A. *The formation $\mathfrak{F}$ is saturated if and only if $G/\Phi_{\mathfrak{F}}(G)\in\mathfrak{F}$ implies $G\in\mathfrak{F}$.*

PROOF. If $\mathfrak{F}$ is saturated, and $G/\Phi_{\mathfrak{F}}(G)\in\mathfrak{F}$, then as remarked above, each supplemented chief factor of $G/\Phi(G)$ is $\mathfrak{F}$-central, and $G/\Phi(G)\in\mathfrak{F}$, so $G\in\mathfrak{F}$. Conversely, since $\Phi_{\mathfrak{F}}(G)\geqq\Phi(G)$, $G/\Phi(G)\in\mathfrak{F}$ implies $G/\Phi_{\mathfrak{F}}(G)\in\mathfrak{F}$, so $G\in\mathfrak{F}$ if the condition holds.

3. Throughout this section, $\mathfrak{F}$ will denote a saturated formation. A subgroup H of G is said to be *$\mathfrak{F}$-immersed in G* if $H\trianglelefteq G$ and each G-chief factor below H is $\mathfrak{F}$-central. The *$\mathfrak{F}$-hypercenter*, $Z_{\mathfrak{F}}(G)$, is defined to be the product in G of all $\mathfrak{F}$-immersed subgroups of G; $Z_{\mathfrak{F}}(G)$ is $\mathfrak{F}$-immersed, and characteristic in G. In the language of [2], $Z_{\mathfrak{F}}(G)$ is the intersection of the $\mathfrak{F}$-normalizers of G, if G is solvable.

PROPOSITION 4. *For a saturated formation $\mathfrak{F}$, $Z_{\mathfrak{F}}(G)\leqq\Phi_{\mathfrak{F}}(G)$.*

PROOF. Let N be a minimal normal subgroup of G, contained in $Z_{\mathfrak{F}}(G)$. Then $Z_{\mathfrak{F}}(G/N)=Z_{\mathfrak{F}}(G)/N$. If N is not contained in $\Phi_{\mathfrak{F}}(G)$, N is complemented by an $\mathfrak{F}$-abnormal maximal subgroup, so N is not $\mathfrak{F}$-central, a contradiction. Thus $N\leqq\Phi_{\mathfrak{F}}(G)$, and $\Phi_{\mathfrak{F}}(G/N)=\Phi_{\mathfrak{F}}(G)/N$. By induction, $Z_{\mathfrak{F}}(G)/N\leqq\Phi_{\mathfrak{F}}(G)/N$, completing our proof.

COROLLARY 5. $Z_{\mathfrak{F}}(G)\cap G_{\mathfrak{F}}\leqq\Phi(G)$.

PROOF. $Z_{\mathfrak{F}}(G)\cap G_{\mathfrak{F}}\leqq\Phi_{\mathfrak{F}}(G)\cap T_{\mathfrak{F}}(G)=\Phi(G)$.

REMARK. The theorem of Gaschütz [4] which Corollary 5 emulates is somewhat sharper, reading $Z_\infty(G)\cap G'\leqq\Phi(G)$. We can recover this by observing that if $\mathfrak{F}^*$ is the smallest saturated formation containing $\mathfrak{F}$, then one can show that $T_{\mathfrak{F}}(G)=T_{\mathfrak{F}^*}(G)$, so that $Z_{\mathfrak{F}^*}(G)\cap G_{\mathfrak{F}}\leqq\Phi(G)$, by the proof given above.

It is reasonable to ask whether or not $\mathfrak{F}$-immersed subgroups are actually in F.

PROPOSITION 6. *For a saturated formation $\mathfrak{F}$, $\mathfrak{F}$-immersed subgroups lie in $\mathfrak{F}$ if and only if $\mathfrak{F}$ is normally closed.*

PROOF. If $H\trianglelefteq G\in\mathfrak{F}$, and $\mathfrak{F}$-immersed subgroups lie in $\mathfrak{F}$, $H\in\mathfrak{F}$. Conversely, suppose H is $\mathfrak{F}$-immersed in G, and $\mathfrak{F}$ is normally closed. By Lemma 5.2 of [3], each $f(p)$ is normally closed. Now if M/N is a G-chief factor below H, $\mathrm{Aut}_H(M/N)$ is isomorphic to a normal subgroup of $\mathrm{Aut}_G(M/N)$, so $\mathrm{Aut}_H(M/N)$ belongs to $f(p)$. If $K\trianglelefteq H$ and $N\leqq K\leqq M$, then $C_H(K/N)$ and $C_H(M/K)$ contain $C_H(M/N)$, so $\mathrm{Aut}_H(K/N)$ and $\mathrm{Aut}_H(M/K)$ are quotients of $\mathrm{Aut}_H(M/N)$ and are in $f(p)$. Thus if we refine the G-chief factors below H to H-chief factors they remain $\mathfrak{F}$-central, and thus $H\in\mathfrak{F}$.

The following technical result is the basis of our study of $\mathfrak{F}$-immersed subgroups.

THEOREM B. *Let H be a solvable normal subgroup of an arbitrary finite group G, and let p be a prime. For any subgroup C of G, if C centralizes each p-chief factor of G between H and $\Phi(H)$, C centralizes each p-chief factor of G below $\Phi(H)$.*

PROOF. We may assume that C is the intersection of the centralizers of the p-chief factors between H and $\Phi(H)$, so that $C\trianglelefteq G$. By a routine induction argument, it is sufficient to consider the following case: K is a minimal normal p-subgroup of G contained in $\Phi(H)$, and C centralizes all p-factors of G between K and H.

For any prime r dividing $|C|$, let R be an r-Sylow subgroup of C. Define $S=HR$ and $T=C\cap S$; S is solvable, and $R\leqq T\trianglelefteq S$. Now set $N=N_S(T\cap U)$, where U is a p'-Hall subgroup of S. Since T normalizes H, H can be regarded as a group with operators, the conjugations from T. Let A/B be a T-chief factor of H. Since $U\leqq N$, $NB\geqq A$ if A/B is not a p-group. If A/B is a p-group, then P. Hall's theory of system normalizers [7] applies. This yields that $A\leqq NB$ if T centralizes A/B, and $A\cap N\leqq B$ otherwise. Since $T\leqq C$ centralizes all T-factors of p-power order above K, $NK\cap H=H=K(N\cap H)=\Phi(H)(N\cap H)=N\cap H$. Thus $H\leqq N$, and T centralizes all T-factors of H contained in K. Thus for some sufficiently long commutator, $[K, R, \cdots, R]=1$. For $r\neq p$, we have $[K, R]=1$ by

Theorem 5.3.2 of [**6**]. Therefore $C_G(K)$ contains the r-Sylow subgroups of C for all primes $r \neq p$. Thus for any p-Sylow subgroup P of C, $C \leqq PC_G(K)$. Now we have $[K, C] \leqq [K, PC_G(K)] = [K, P] < K$. Since C is normal and K is minimal normal, we have $[K, C] = 1$, completing our proof.

Our next theorem extends the main result of [**1**].

THEOREM C. *Let $\mathfrak{F}$ be a saturated formation. If H is a solvable normal subgroup of an arbitrary finite group G and $H/\Phi(H)$ is $\mathfrak{F}$-immersed in $G/\Phi(H)$, then H is $\mathfrak{F}$-immersed in G.*

PROOF. By induction, we need only study the case where M is minimal normal in G, $M \leqq \Phi(H)$, and H/M is $\mathfrak{F}$-immersed in G/M. Since $M \leqq \Phi(H)$, M is a p-group for some prime p. Let C be the intersection of the centralizers of the p-chief factors of G between M and H. Since these factors are each $\mathfrak{F}$-central, $G/C \in f(p)$. By Theorem B, C centralizes M, so $\mathrm{Aut}_G(M)$ is a quotient of G/C and therefore also lies in $f(p)$. By definition, M is an $\mathfrak{F}$-central factor, and H is $\mathfrak{F}$-immersed as claimed.

For the formation of nilpotent groups, $\mathfrak{F}$-central factors (*alias* central factors) can be detected by use of commutators; we now generalize this to other saturated formations. For $H \trianglelefteq G$, define the *$\mathfrak{F}$-hypocenter of H relative to G* to be the smallest G-normal subgroup K contained in H so that all G-chief factors between K and H are $\mathfrak{F}$-central; we write $[H, G; \mathfrak{F}]$ for this subgroup.

PROPOSITION 7. *Let $\mathfrak{F}$ be a saturated formation.*

(a) *A G-chief factor H/K is $\mathfrak{F}$-central if $[H, G; \mathfrak{F}] \leqq K$.*

(b) $[G, G; \mathfrak{F}] = G_{\mathfrak{F}}$.

(c) *If $\mathfrak{F}$ is normally closed and $H \trianglelefteq G$, then $[H, G; \mathfrak{F}] \leqq G_{\mathfrak{F}} \cap H$.*

PROOF. The first two parts are trivial; for (c) we note that the G-chief factors from $G_{\mathfrak{F}} \cap H$ to H are G-isomorphic to those from $HG_{\mathfrak{F}}$ to $HG_{\mathfrak{F}}/G_{\mathfrak{F}}$, which are $\mathfrak{F}$-central with respect to G; apply Proposition 6.

We now give an analogue of a result on Φ-free groups due to Gaschütz.

THEOREM D. *Let $\mathfrak{F}$ be a saturated, normally closed formation. Then if $\Phi(G) = 1$, $Z_{\mathfrak{F}}(G) = \Phi_{\mathfrak{F}}(G)$.*

PROOF. $[\Phi_{\mathfrak{F}}(G), G; \mathfrak{F}] \leqq \Phi_{\mathfrak{F}}(G) \cap G_{\mathfrak{F}} \leqq \Phi_{\mathfrak{F}}(G) \cap T_{\mathfrak{F}}(G) = \Phi(G) = 1$. Thus $\Phi_{\mathfrak{F}}(G)$ is $\mathfrak{F}$-hypercentral, so $Z_{\mathfrak{F}}(G) = \Phi_{\mathfrak{F}}(G)$.

4. Gaschütz showed in [**5**] that a saturated formation $\mathfrak{F}$ containing all nilpotent groups has the following property: for a normal subgroup H of G, if $H/(H \cap \Phi(G))$ is in $\mathfrak{F}$, then H belongs to $\mathfrak{F}$. It is easy to see that this property characterizes such formations. We now exhibit a stronger property which characterizes the saturated, normally closed formations containing the nilpotent groups.

THEOREM E. *For a formation $\mathfrak{F}$, the following properties are equivalent:*
(a) *$\mathfrak{F}$ is normally closed, saturated, and contains all nilpotent groups.*
(b) *If $H \trianglelefteq G$ and $H/(H \cap \Phi_{\mathfrak{F}}(G))$ belongs to $\mathfrak{F}$, then H belongs to $\mathfrak{F}$.*

PROOF. Assume (a), and suppose N is a minimal normal subgroup of G contained in $H \cap \Phi_{\mathfrak{F}}(G)$ and $H/N \in \mathfrak{F}$. If $N \leqq \Phi(G)$, $H \in \mathfrak{F}$ by the Theorem of Gaschütz cited above. If $N \nleqq \Phi(G)$, $N \cap \Phi(G) = 1$, and by Theorem D, N is $\mathfrak{F}$-central. By the proof of Proposition 6, N refines to $\mathfrak{F}$-central H-factors, and therefore $H \in \mathfrak{F}$.

Now assume (b); if $G \in \mathfrak{F}$, G has no $\mathfrak{F}$-abnormal maximal subgroups, so $\Phi_{\mathfrak{F}}(G) = G$, and for all $H \trianglelefteq G$, $H/H \cap \Phi_{\mathfrak{F}}(G) = 1$, so $H \in \mathfrak{F}$. With $H = G$, (b) becomes the condition for saturation given in Theorem A. The $\mathfrak{F}$-Frattini subgroup of a cyclic group of order p^2 is at least order p, so $\mathfrak{F}$ contains all groups of prime order and therefore all nilpotent groups.

COROLLARY 8. *If $\mathfrak{F}$ satisfies condition* (a) *of Theorem* E, *$\Phi_{\mathfrak{F}}(G)$ belongs to $\mathfrak{F}$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601