ON $\mathfrak{g}$-ABNORMAL MAXIMAL SUBGROUPS
OF A FINITE SOLVABLE GROUP

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Abstract. Let $\Delta(G)$ be the intersection of the nonnormal maximal subgroup of a finite group. W. Gaschütz has shown that $\Delta(G)$ is nilpotent and that $\Delta(G)/\Phi(G)$ is the center of $G/\Phi(G)$. This note, by considering the intersection of the $\mathfrak{g}$-abnormal maximal subgroups, generalizes these results for a saturated formation $\mathfrak{g}$.

In [2] Gaschütz shows that the intersection of the nonnormal maximal subgroups of the group $G$, $\Delta(G)$, is nilpotent and $\Delta(G)/\Phi(G)=Z(G/\Phi(G))$. In the theory of saturated formations the $\mathfrak{g}$-abnormal maximal subgroups, introduced by Carter and Hawkes [1], play a role similar to that classically played by the nonnormal maximal subgroups. By considering the intersection of the $\mathfrak{g}$-abnormal maximal subgroups of a finite group $G$, this note provides a generalization of these results of Gaschütz.

All groups considered in this note are finite and solvable. The notation and terminology is standard. Let $G$ be a finite solvable group and $\mathfrak{g} = \{\mathfrak{g}(p)\}$ an integrated (i.e. $\mathfrak{g} \supseteq \mathfrak{g}(p)$) local formation. $\Phi(G)$ denotes the Frattini subgroup of $G$ and $G_{\mathfrak{g}}$ denotes the $\mathfrak{g}$-residual of $G$ (i.e. the smallest normal subgroup of $G$ such that $G_{\mathfrak{g}}G = G$).

In [1] Carter and Hawkes call a $\mathfrak{g}$-chief-factor $H/K$ of $G$ $\mathfrak{g}$-central if $G/C_{G}(H/K) \in \mathfrak{g}(p)$; otherwise $H/K$ is called $\mathfrak{g}$-eccentric. They call a maximal subgroup $M$ of $G$ $\mathfrak{g}$-normal if $M/\text{Core}(M) \in \mathfrak{g}(p)$ where $p$ is the prime dividing $[G:M]$; otherwise $M$ is called $\mathfrak{g}$-abnormal. It is shown [1, Lemma 2.3] that a maximal subgroup $M$ of $G$ is $\mathfrak{g}$-normal if and only if it complements an $\mathfrak{g}$-central chief factor. We note that although in [1] it is assumed all $\mathfrak{g}(p)$ are nonempty, this condition is not necessary for the above definitions and result.

In [3] Huppert defined the $\mathfrak{g}$-hypercenter of $G$, $Z_{\mathfrak{g}}(G)$, to be the largest normal subgroup of $G$ such that all chief factors of $G$ below $Z_{\mathfrak{g}}(G)$ are $\mathfrak{g}$-central. He shows [3, Satz 1.5] that $Z_{\mathfrak{g}}(G) \in \mathfrak{g}$ whenever the $\mathfrak{g}(p)$ are normal subgroup closed.

Lemma 1. Let $M$ be a maximal subgroup of $G$. If $M$ is $\mathfrak{g}$-abnormal, then $M \supseteq Z_{\mathfrak{g}}(G)$. If $M$ is $\mathfrak{g}$-normal, then $M \supseteq G_{\mathfrak{g}}$.
Proof. Let $M$ be $\mathfrak{g}$-abnormal and suppose $Z\mathfrak{g}(G)M = G$. Let $R = \text{Core}(M)$ and $S = R \cap Z\mathfrak{g}(G)$. Let $T \leq Z\mathfrak{g}(G)$, such that $T/S$ is a chief factor of $G$. $T/S$ is $\mathfrak{g}$-central and complemented by $M$. Hence $M$ is $\mathfrak{g}$-normal. $M$ cannot be both $\mathfrak{g}$-normal and $\mathfrak{g}$-abnormal, therefore $M \leq Z\mathfrak{g}(G)$.

Let $M$ be $\mathfrak{g}$-normal and suppose $G\mathfrak{g}M = G$. Let $R = \text{Core}(M)$. $G/R = (M/R)(S/R)$ where $S/R$ is a unique self-centralizing minimal normal subgroup of $G/R$. If $[G:M] = p^a$, then $S/R$ is an $\mathfrak{g}$-central $p$-chief factor of $G$, and $G/S \in \mathfrak{g}(p)$. Since $G \supseteq \mathfrak{g}(p)$, $G/S \in \mathfrak{g}$ and $S \supseteq G\mathfrak{g}$. By the Dedekind property $S = G\mathfrak{g}(M \cap S) = G\mathfrak{g}R$. Thus $S/R$ is $G$-isomorphic to $G\mathfrak{g}/R \cap G\mathfrak{g}$ and $G\mathfrak{g}/R \cap G\mathfrak{g}$ is $\mathfrak{g}$-central. This is impossible; therefore $M \subseteq G\mathfrak{g}$.

We denote the intersection of all $\mathfrak{g}$-abnormal maximal subgroups of $G$ by $\Delta\mathfrak{g}(G)$.

Theorem 1. The following statements are valid.

(a) $\Delta\mathfrak{g}(G) \cap G\mathfrak{g} \subseteq \Phi(G)$.

(b) $\Delta\mathfrak{g}(G) \supseteq Z\mathfrak{g}(G)$.

(c) $\Delta\mathfrak{g}(G)/(\Phi(G)) = Z\mathfrak{g}(G/\Phi(G))$.

Proof. Statements (a) and (b) are an immediate consequence of Lemma 1, thus we need only show the validity of statement (c).

As $\Delta\mathfrak{g}(G)/(\Phi(G)) = \Delta\mathfrak{g}(G)/(\Phi(G))$, it follows from statement (b) that $\Delta\mathfrak{g}(G)/(\Phi(G)) \supseteq Z\mathfrak{g}(G/\Phi(G))$. Therefore $\Delta\mathfrak{g}(G)/(\Phi(G)) = Z\mathfrak{g}(G/\Phi(G))$. Thus statement (a) implies that $\Delta\mathfrak{g}(G)/(\Phi(G)) \cap (G/\Phi(G))_{\mathfrak{g}} = \{1\}$. If $H/K$ is a chief factor of $G/\Phi(G)$ which lies below $\Delta\mathfrak{g}(G)/(\Phi(G))$, then $H/(G/\Phi(G))_{\mathfrak{g}}[K/(G/\Phi(G))]_{\mathfrak{g}}$ is $G$-isomorphic to $H/K$. Since $H/(G/\Phi(G))_{\mathfrak{g}}[K/(G/\Phi(G))]_{\mathfrak{g}}$ lies above $(G/\Phi(G))_{\mathfrak{g}}$, it is $\mathfrak{g}$-central and thus $H/K$ is $\mathfrak{g}$-central. Hence all chief factors of $G/(\Phi(G))$ lying below $\Delta\mathfrak{g}(G)/(\Phi(G))$ are $\mathfrak{g}$-central so that $\Delta\mathfrak{g}(G)/(\Phi(G)) \leq Z\mathfrak{g}(G/\Phi(G))$. Therefore $\Delta\mathfrak{g}(G)/(\Phi(G)) = Z\mathfrak{g}(G/\Phi(G))$.

From statements (a) and (b) above, we see

Corollary. $Z\mathfrak{g}(G) \cap G\mathfrak{g} \subseteq \Phi(G)$.

We now investigate more closely the structure of $\Delta\mathfrak{g}(G)$. For this we let $\pi$ be the set of primes for which the $\mathfrak{g}(p)$ are nonempty. It is useful to present two elementary lemmas. The first of these is a consequence of the conjugacy of $\mathfrak{g}$-projectors; its proof is omitted.

Lemma 2. Let $H$ be a normal subgroup of $G$ and $E$ an $\mathfrak{g}$-projector of $H$, then $G = HN$ where $N = N\mathfrak{g}(E)$.

Lemma 3. Let $H$ be a normal $\pi$-subgroup of $G$ with $K \leq H$ such that $K \leq \Phi(G) \cap H$. If $H/K \in \mathfrak{g}$, then $H \in \mathfrak{g}$.

Proof. Let $E$ be an $\mathfrak{g}$-projector of $H$. $H = EK$ and, by Lemma 2, $G = HN\mathfrak{g}(E) = KN\mathfrak{g}(E) = \Phi(G)N\mathfrak{g}(E) = N\mathfrak{g}(E)$. Hence $E \leq H$. Since $H$ is a $\pi$-group and $\mathfrak{g}(p) \neq \emptyset$ for $p \in \pi$, $N\mathfrak{g}(E) = E$. Therefore $E = H$ and $H \in \mathfrak{g}$.
Theorem 2. $\Delta_3(G) = P \triangleleft Q$ where:
1. $P$ is a $\pi$-group with $P \geq Z_3(G)$ and $P/P \cap \Phi(G) \cong Z_3(G/\Phi(G))$; and
2. $Q$ is a $\pi'$-group with $Q \leq \Phi(G)$.

Furthermore if the $\mathfrak{F}(p)$ are normal subgroup closed, then $P \in \mathfrak{F}$.

Proof. Since the $\mathfrak{F}$-central chief factors are necessarily $\pi$-groups, $Z_3(G/\Phi(G))$ is a $\pi$-group. By Theorem 1, $\Delta_3(G)/\Phi(G)$ is then a $\pi$-group. Hence from the nilpotence of $\Phi(G)$ we conclude that $\Delta_3(G) = PQ$ where $Q$ is a normal Hall $\pi'$-subgroup of $\Delta_3(G)$ with $Q \leq \Phi(G)$ and $P$ is a Hall $\pi$-subgroup of $\Delta_3(G)$. By the Frattini argument $G = N_G(P)\Delta_3(G) = N_G(P)Q = N_G(P)$ so that $P$ is a normal subgroup of $G$. Thus $\Delta_3(G) = P \triangleleft Q$.

Since $Z_3(G)$ is a $\pi$-group and $\Delta_3(G) \geq Z_3(G)$, $P \geq Z_3(G)$, $Q \leq \Phi(G)$ implies that $\Delta_3(G)/\Phi(G) = P/P \cap \Phi(G)$, thus by Theorem 1, $P/P \cap \Phi(G) = Z_3(G/\Phi(G))$.

If the $\mathfrak{F}(p)$ are normal subgroup closed, then by Huppert’s result $Z_3(G/\Phi(G)) \in \mathfrak{F}$. Hence $P/P \cap \Phi(G) \in \mathfrak{F}$ and by Lemma 3 we conclude that $P \in \mathfrak{F}$.

Corollary. If for all $p$, the $\mathfrak{F}(p)$ are nonempty and normal subgroup closed, then $\Delta_3(G) \in \mathfrak{F}$.

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References