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ON $\mathfrak{F}$-ABNORMAL MAXIMAL SUBGROUPS OF A FINITE SOLVABLE GROUP

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ABSTRACT. Let $\Delta(G)$ be the intersection of the nonnormal maximal subgroup of a finite group. W. Gaschütz has shown that $\Delta(G)$ is nilpotent and that $\Delta(G)/\Phi(G)$ is the center of $G/\Phi(G)$. This note, by considering the intersection of the $\mathfrak{F}$-abnormal maximal subgroups, generalizes these results for a saturated formation $\mathfrak{F}$.

In [2] Gaschütz shows that the intersection of the nonnormal maximal subgroups of the group G, $\Delta(G)$, is nilpotent and $\Delta(G)/\Phi(G)=Z(G/\Phi(G))$. In the theory of saturated formations the $\mathfrak{F}$-abnormal maximal subgroups, introduced by Carter and Hawkes [1], play a role similar to that classically played by the nonnormal maximal subgroups. By considering the intersection of the $\mathfrak{F}$-abnormal maximal subgroups of a finite group G, this note provides a generalization of these results of Gaschütz.

All groups considered in this note are finite and solvable. The notation and terminology is standard. Let G be a finite solvable group and $\mathfrak{F}=\{\mathfrak{F}(p)\}$ an integrated (i.e. $\mathfrak{F}\supseteq\mathfrak{F}(p)$) local formation. $\Phi(G)$ denotes the Frattini subgroup of G and $G_{\mathfrak{F}}$ denotes the $\mathfrak{F}$-*residual* of G (i.e. the smallest normal subgroup of G such that $G/G_{\mathfrak{F}}\in\mathfrak{F}$).

In [1] Carter and Hawkes call a p-chief factor H/K of G $\mathfrak{F}$-*central* if $G/C_G(H/K)\in\mathfrak{F}(p)$; otherwise H/K is called $\mathfrak{F}$-*eccentric*. They call a maximal subgroup M of G $\mathfrak{F}$-*normal* if $M/\mathrm{Core}(M)\in\mathfrak{F}(p)$ where p is the prime dividing $[G:M]$; otherwise M is called $\mathfrak{F}$-*abnormal*. It is shown [1, Lemma 2.3] that a maximal subgroup M of G is $\mathfrak{F}$-normal if and only if it complements an $\mathfrak{F}$-central chief factor. We note that although in [1] it is assumed all $\mathfrak{F}(p)$ are nonempty, this condition is not necessary for the above definitions and result.

In [3] Huppert defined the $\mathfrak{F}$-*hypercenter* of G, $Z_{\mathfrak{F}}(G)$, to be the largest normal subgroup of G such that all chief factors of G below $Z_{\mathfrak{F}}(G)$ are $\mathfrak{F}$-central. He shows [3, Satz 1.5] that $Z_{\mathfrak{F}}(G)\in\mathfrak{F}$ whenever the $\mathfrak{F}(p)$ are normal subgroup closed.

LEMMA 1. *Let M be a maximal subgroup of G. If M is $\mathfrak{F}$-abnormal, then $M\geqq Z_{\mathfrak{F}}(G)$. If M is $\mathfrak{F}$-normal, then $M\geqq G_{\mathfrak{F}}$.*

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PROOF. Let M be $\mathfrak{F}$-abnormal and suppose $Z_{\mathfrak{F}}(G)M=G$. Let $R=\text{Core}(M)$ and $S=R\cap Z_{\mathfrak{F}}(G)$. Let $T\leqq Z_{\mathfrak{F}}(G)$, such that T/S is a chief factor of G. T/S is $\mathfrak{F}$-central and complemented by M. Hence M is $\mathfrak{F}$-normal. M cannot be both $\mathfrak{F}$-normal and $\mathfrak{F}$-abnormal, therefore $M\geqq Z_{\mathfrak{F}}(G)$.

Let M be $\mathfrak{F}$-normal and suppose $G_{\mathfrak{F}}M=G$. Let $R=\text{Core}(M)$. $G/R=(M/R)(S/R)$ where S/R is a unique self-centralizing minimal normal subgroup of G/R. If $[G:M]=p^a$, then S/R is an $\mathfrak{F}$-central p-chief factor of G, and $G/S\in\mathfrak{F}(p)$. Since $\mathfrak{F}\supseteq\mathfrak{F}(p)$, $G/S\in\mathfrak{F}$ and $S\geqq G_{\mathfrak{F}}$. By the Dedekind property $S=G_{\mathfrak{F}}(M\cap S)=G_{\mathfrak{F}}R$. Thus S/R is G-isomorphic to $G_{\mathfrak{F}}/R\cap G_{\mathfrak{F}}$ and $G_{\mathfrak{F}}/R\cap G_{\mathfrak{F}}$ is $\mathfrak{F}$-central. This is impossible; therefore $M\geqq G_{\mathfrak{F}}$.

We denote the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of G by $\Delta_{\mathfrak{F}}(G)$.

THEOREM 1. *The following statements are valid.*

(a) $\Delta_{\mathfrak{F}}(G)\cap G_{\mathfrak{F}}\leqq\Phi(G)$.

(b) $\Delta_{\mathfrak{F}}(G)\geqq Z_{\mathfrak{F}}(G)$.

(c) $\Delta_{\mathfrak{F}}(G)/\Phi(G)=Z_{\mathfrak{F}}(G/\Phi(G))$.

PROOF. Statements (a) and (b) are an immediate consequence of Lemma 1, thus we need only show the validity of statement (c).

As $\Delta_{\mathfrak{F}}(G/\Phi(G))=\Delta_{\mathfrak{F}}(G)/\Phi(G)$, it follows from statement (b) that $\Delta_{\mathfrak{F}}(G)/\Phi(G)\geqq Z_{\mathfrak{F}}(G/\Phi(G))$. $(G/\Phi(G))_{\mathfrak{F}}=G_{\mathfrak{F}}\Phi(G)/\Phi(G)$, thus statement (a) implies that $\Delta_{\mathfrak{F}}(G)/\Phi(G)\cap(G/\Phi(G))_{\mathfrak{F}}=\{1\}$. If H/K is a chief factor of $G/\Phi(G)$ which lies below $\Delta_{\mathfrak{F}}(G)/\Phi(G)$, then $H(G/\Phi(G))_{\mathfrak{F}}/K(G/\Phi(G))_{\mathfrak{F}}$ is G-isomorphic to H/K. Since $H(G/\Phi(G))_{\mathfrak{F}}/K(G/\Phi(G))_{\mathfrak{F}}$ lies above $(G/\Phi(G))_{\mathfrak{F}}$, it is $\mathfrak{F}$-central and thus H/K is $\mathfrak{F}$-central. Hence all chief factors of $G/\Phi(G)$ lying below $\Delta_{\mathfrak{F}}(G)/\Phi(G)$ are $\mathfrak{F}$-central so that $\Delta_{\mathfrak{F}}(G)/\Phi(G)\leqq Z_{\mathfrak{F}}(G/\Phi(G))$. Therefore $\Delta_{\mathfrak{F}}(G)/\Phi(G)=Z_{\mathfrak{F}}(G/\Phi(G))$.

From statements (a) and (b) above, we see

COROLLARY. $Z_{\mathfrak{F}}(G)\cap G_{\mathfrak{F}}\leqq\Phi(G)$.

We now investigate more closely the structure of $\Delta_{\mathfrak{F}}(G)$. For this we let π be the set of primes for which the $\mathfrak{F}(p)$ are nonempty. It is useful to present two elementary lemmas. The first of these is a consequence of the conjugacy of $\mathfrak{F}$-projectors; its proof is omitted.

LEMMA 2. *Let H be a normal subgroup of G and E an $\mathfrak{F}$-projector of H, then $G=HN$ where $N=N_G(E)$.*

LEMMA 3. *Let H be a normal π-subgroup of G with $K\trianglelefteq H$ such that $K\leqq\Phi(G)\cap H$. If $H/K\in\mathfrak{F}$, then $H\in\mathfrak{F}$.*

PROOF. Let E be an $\mathfrak{F}$-projector of H. $H=EK$ and, by Lemma 2, $G=HN_G(E)=KN_G(E)=\Phi(G)N_G(E)=N_G(E)$. Hence $E\trianglelefteq H$. Since H is a π-group and $\mathfrak{F}(p)\neq\varnothing$ for $p\in\pi$, $N_H(E)=E$. Therefore $E=H$ and $H\in\mathfrak{F}$.

THEOREM 2. $\Delta_{\mathfrak{F}}(G)=P\oplus Q$ *where*:

1. *P is a π-group with $P\geqq Z_{\mathfrak{F}}(G)$ and $P/P\cap\Phi(G)\cong Z_{\mathfrak{F}}(G/\Phi(G))$; and*
2. *Q is a π'-group with $Q\leqq\Phi(G)$.*

Furthermore if the $\mathfrak{F}(p)$ are normal subgroup closed, then $P\in\mathfrak{F}$.

PROOF. Since the $\mathfrak{F}$-central chief factors are necessarily π-groups, $Z_{\mathfrak{F}}(G/\Phi(G))$ is a π-group. By Theorem 1, $\Delta_{\mathfrak{F}}(G)/\Phi(G)$ is then a π-group. Hence from the nilpotence of $\Phi(G)$ we conclude that $\Delta_{\mathfrak{F}}(G)=PQ$ where Q is a normal Hall π'-subgroup of $\Delta_{\mathfrak{F}}(G)$ with $Q\leqq\Phi(G)$ and P is a Hall π-subgroup of $\Delta_{\mathfrak{F}}(G)$. By the Frattini argument $G=N_G(P)\Delta_{\mathfrak{F}}(G)=N_G(P)Q=N_G(P)$ so that P is a normal subgroup of G. Thus $\Delta_{\mathfrak{F}}(G)=P\oplus Q$.

Since $Z_{\mathfrak{F}}(G)$ is a π-group and $\Delta_{\mathfrak{F}}(G)\geqq Z_{\mathfrak{F}}(G)$, $P\geqq Z_{\mathfrak{F}}(G)$. $Q\leqq\Phi(G)$ implies that $\Delta_{\mathfrak{F}}(G)/\Phi(G)=P/P\cap\Phi(G)$, thus by Theorem 1, $P/P\cap\Phi(G)=Z_{\mathfrak{F}}(G/\Phi(G))$.

If the $\mathfrak{F}(p)$ are normal subgroup closed, then by Huppert's result $Z_{\mathfrak{F}}(G/\Phi(G))\in\mathfrak{F}$. Hence $P/P\cap\Phi(G)\in\mathfrak{F}$ and by Lemma 3 we conclude that $P\in\mathfrak{F}$.

COROLLARY. *If for all p, the $\mathfrak{F}(p)$ are nonempty and normal subgroup closed, then $\Delta_{\mathfrak{F}}(G)\in\mathfrak{F}$.*

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