PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

RINGS WITH INVOLUTION WHOSE SYMMETRIC ELEMENTS ARE REGULAR

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ABSTRACT. In this work we determine the structure of 2-torsion-free rings with involution whose nonzero symmetric elements do not annihilate one another.

In this work we obtain a rather complete structure theorem for rings with involution whose symmetric elements are not zero divisors. As a corollary we have a theorem of Osborn [2] concerning simple rings whose symmetric elements are invertible.

R will always denote an associative ring with involution, denoted by $*$. That is, $*$ is an anti-automorphism of R of period two. Let $S=\{r \in R \mid r^*=r\}$ be the set of symmetric elements of R and $K=\{r \in R \mid r^*=-r\}$ the set of skew-symmetric elements. We shall assume that R is 2-torsion-free, and so $S\cap K=0$ and $2R\subset S+K$ since $2r=(r+r^*)+(r-r^*)$. Let $S'=S-\{0\}$. We shall say that S' has no zero divisors to mean: (1) S' is not empty; and (2) given $s, t \in S'$, then $st\neq 0$. Note that if S' were allowed to be empty then every element of R has square zero, and so $R^3=0$. Lastly, for $a, b \in R$ let $[a, b]=ab-ba$.

The first goal is to show that if S' has no zero divisors then R has a unique maximal nilpotent ideal. In fact we can assume somewhat less to begin with.

LEMMA 1. *If $x \in R$, $2x=s+k$ and $xx^*=x^*x=0$, then $s^2=k^2$ and $sk=ks$.*

PROOF. Clearly $(2x)(2x)^*=(2x)^*(2x)=0$. Thus $(s+k)(s-k)=0=(s-k)(s+k)$. Hence $s^2-k^2+ks-sk=0=s^2-k^2+sk-ks$, and so, $2(sk-ks)=0$. Since R is 2-torsion-free, $sk-ks=0$, and thus $s^2=k^2$.

LEMMA 2. *If S' has no nilpotent elements then every nil right (left) ideal N of R satisfies*

(i) $N^3=0$;

(ii) $N\subset K$;

(iii) *if $x \in N$ then $x^2=0$.*

Received by the editors September 20, 1971.

AMS 1970 *subject classifications*. Primary 16A28; Secondary 16A48.

Key words and phrases. Ring with involution, regular symmetric elements.

[1] This work was supported by NSF Grant GP-29119X.

PROOF. If $x \in N$ then $xx^* \in N \cap S$. Since S' has no nilpotent elements, $xx^*=0$. But $(x^*x)^2=0$ so $x^*x=0$ as well. By Lemma 1, if $2x=s+k$ then $s^2=k^2$ and $sk=ks$. Suppose $x^2=0$. Then $4x^2=0$ and $0=s^2+k^2+2sk=2(s^2+sk)$. Now $sk \in K$ so $s^2=0$. But S' has no nilpotent elements, so $s=0$. Thus $2x$ and so $x \in K$. Next suppose that $x \in N$ and $x^2 \in K$. Then $4x^2 \in K$. If $2x=s+k$, this says $s^2+k^2+2sk \in K$. But now $s^2+k^2=2s^2 \in K$. So $2s^2 \in K \cap S$ and as above $s=0$, so $x \in K$. We claim that $N \subset K$. If $x \in N$ then $x^m=0$ for some m. If $t \geqq m/2$ then $(x^t)^2=0$. Since $x^t \in N$, by what we have shown above, $x^t \in K$. Let t be minimal with $x^{t+p} \in K$ for $p \geqq 0$. If $t>1$ then $(x^{t-1})^2=x^{t+(t-2)}$, so $(x^{t-1})^2 \in K$ which we have seen forces $x^{t-1} \in K$ contradicting the minimality of t. Thus $t=1$ and so $N \subset K$. But $x^m=0$ implies $(x^2)^m=0$. Since $x^2 \in S$ we have $x^2=0$ for all $x \in N$. As R is 2-torsion-free any subring having the square of every element zero is nilpotent of index 3. ■

If R is a ring with involution and A is an ideal of R with $A^*=A$ then the quotient ring R/A inherits an involution defined by $(r+A)^*=r^*+A$. If $x+A$ is symmetric in R/A then $x-x^* \in A$. Since $2x=x+x^*+(x-x^*)$ we have $2x+A=(x+x^*)+A$. Thus if R/A is 2-torsion-free, twice every symmetric element comes from a symmetric element in R. We use this observation together with Lemma 2 to reduce to the case in which R has no nil ideals.

THEOREM 3. *Suppose S' has no zero divisors. Then R has a unique maximal nilpotent ideal N satisfying*:

(i) $N \subset K$;

(ii) $N^3=0$;

(iii) *if* $x \in N$ *then* $x^2=0$;

(iv) N *contains all nil one-sided ideals of* R;

(v) R/N *is a 2-torsion-free ring with involution containing no nil ideals*;

(vi) *If* $S(R/N)$ *are the symmetric elements of* R/N *then* $S'(R/N)$ *has no zero divisors.*

PROOF. Let N be the sum of all nilpotent ideals of R. By Lemma 2, (i)–(iv) hold, and N is the unique maximal nilpotent ideal of R. As $N^*=N$ we know that R/N carries an induced involution and clearly contains no nil ideals. Let $T=\{r \in R \mid 2r \in N\}$. T is an ideal of R. If $x \in T$ then $(2x)^m=2^m x^m=0$. Since R is 2-torsion-free $x^m=0$. Hence T is nil, and so, $T \subset N$. Thus R/N is 2-torsion-free. Suppose x_1+N, $x_2+N \in S'(R/N)$ and $x_1x_2 \in N$. Since $2x_1+N$, $2x_2+N \in S'(R/N)$ and $4x_1x_2 \in N$, we may assume by the discussion above that x_1, $x_2 \in S'$. But $x_1x_2 \in N$ says $(x_1x_2)^2=0$. Since $N \subset K$, $-x_1x_2=(x_1x_2)^*=x_2x_1$, so $0=x_1x_2x_1x_2=-x_1^2x_2^2$. Thus $x_1^2=0$ or $x_2^2=0$, and so, $x_1=0$ or $x_2=0$.

THEOREM 4. *Let R be semiprime and suppose S' has no zero divisors. If $y \in R$ is a zero divisor in R then $yy^*=y^*y=0$. In particular every element of S' is regular in R.*

PROOF. Let us first note that if $r^2=0$ then $r^*rrr^*=0$, so our assumption on S' forces, say, $r^*r=0$. But $(rr^*)^2=0$, and so $0=r^*r=rr^*$. Using Lemma 1, as in Lemma 2, we conclude that $r \in K$. Suppose now that $xy=0$ but $yy^* \neq 0$. Since $(yRx)^2=0$ we have $yRx \subset K$. Thus if $r \in R$, $-yrx=x^*r^*y^*$, which implies $x^*r^*y^*y=0$. Also $(y^2Rx)^2=0$ and so we obtain $x^*r^*(y^*)^2y^2=0$. Thus Ry^*yx^* and $R(y^*)^2y^2x^*$ are nil left ideals of R of index 2. By Levitzki's Theorem [1, Lemma 1.1] these ideals are zero, since R is semiprime. Thus $y^*yx^*=(y^*)^2y^2x^*=0$. Now $xy=0$ says $(yx)^2=0$, so $yx \in K$. Therefore $-yx=x^*y^*$. Using $y^*yx^*=0$ we have $y^*yx^*y^*=0$, and so $-y^*y^2x=0$. Together with $(y^*)^2y^2x^*=0$ we obtain $(y^*)^2y^2(x+x^*)=0$. Since $(y^*)^2y^2 \in S$ either $x+x^*=0$ or $(y^*)^2y^2=0$. In the latter case $(yy^*)(y^*y)(yy^*)(y^*y)=0$. Since $(yy^*y^*y)^2=0$ it is in K, so we obtain $(yy^*)^2(y^*y)^2=0$. Hence $yy^*=0$ or $y^*y=0$, both of which are impossible. Thus we must have $x+x^*=0$, so $x \in K$. Since $x^2yy^*=0$ we have $x^2=0$.

From above we have $x^*Ry^*y=0$, and so $xRy^*y=0$. But $(y^*y)(y^*y)^* \neq 0$ so $xR \subset K$. Hence $-xr=-r^*x$ for all $r \in R$. Thus $xrxt=r^*x^2t=0$. Since R is semiprime and $(xR)^2=0$ we must have $x=0$. Thus if $xy=0$ for $x, y \neq 0$ we must have $yy^*=0$, and so $y^*y=0$. If $yx=0$ to begin with then $x^*y^*=0$, so again $y^*y=yy^*=0$.

LEMMA 5. *If R is semiprime and S' has no zero divisors then either $S \subset Z$, the center of R, or R has no nilpotent elements.*

PROOF. As in Theorem 4 if $x \in R$ and $x^2=0$ then $x \in K$. Further $(xRx)^2=0$, so $xRx \subset K$. As $x \in K$ we have $xSx \subset K \cap S=0$. If $s \in S$ then $(xs)^2=0$ says $xs \in K$, so $-xs=-sx$ and x commutes elementwise with S. Suppose $S \not\subset Z$. Then $2S \not\subset Z$. Let $(2S)^-$ be the subring of $2R$ generated by $2S$. Since $2R \subset S+K$, by an easy argument [1, p. 10] $(2S)^-$ is a Lie ideal of $2R$. Since $2R$ is a 2-torsion-free semiprime ring and $2S$ is not in its center we have that $(2S)^-$ must contain a nonzero ideal of $2R$ [1, Lemma 1.3]. Let I be the sum of all such ideals. Clearly $I^*=I$, and also, since x commutes with S, x commutes elementwise with I. Thus $(2x)I2x=0$ and $2xI$ is a nilpotent ideal in $2R$, which is impossible. Hence $xI=0$. If $I \cap S \neq 0$ then $x=0$ since Theorem 4 implies that elements of S' are regular in R. Assume then that $I \cap S=0$. Since $I^*=I$, for $y \in I$, $y+y^* \in I \cap S=0$. Thus $I \subset K$. But now if $y \in I$, $y^2 \in I \cap S=0$. Since I is nil of index 2, its cube is zero, contradicting $2R$ a semiprime ring. The net result is that R can contain no elements of square zero, and so, no nilpotent elements.

THEOREM 6. *Let R be semiprime and suppose S' has no zero divisors. Then one of the following holds*:

(i) *R is a domain*;

(ii) *$S' \subset Z$, the center of R, and $R(S')^{-1}$, the localization of R at S', is the complete 2×2 matrix ring over a field*;

(iii) *R is a subring of $A \oplus A^{\mathrm{op}}$, where $A \simeq R/P$ is a domain, A^{op} is its opposite ring, $P \cup P^*$ is the set of zero divisors of R, and $*$ in R is induced by interchanging co-ordinates in $A\oplus A^{\mathrm{op}}$.*

PROOF. By Lemma 5 either $S' \subset Z$ or R has no nilpotent elements. If R is commutative, then since R is semiprime, it has no nilpotent elements. Thus we may consider two cases: one where $S' \subset Z$ and R is not commutative, and the other where R has no nonzero nilpotent elements.

To start with, suppose $S' \subset Z$ and R is not commutative. Since $S \subset Z$ we have $[R, R] \subset K$. Suppose $I \neq 0$ is an ideal of R consisting only of zero divisors. Note that by Theorem 4 there is no distinction between left, right, and two-sided zero divisors. If $I \not\subset Z$ then $[I, R] \neq 0$. Now $[I, R] \subset I \cap K$, so it consists of zero divisors in K. If $x \in [I, R]$ then $x^2 \in S$, so $x^2=0$, using Theorem 4 again. But for any $r \in R$, $xr-rx \in [I, R]$, so $0=(xr-rx)^2x$. Thus xR is a nil right ideal of R of index 3. Since R is semiprime, $x=0$. Hence if I consists of zero divisors, $I \subset Z$. Now suppose that R is not prime. Let P be an ideal of R maximal with respect to exclusion of the multiplicative semigroup of regular elements of R. By the usual argument, P is prime and so $P \neq 0$ since R is not prime. Since P consists of zero divisors $P \subset Z$ by what was just shown. Hence $[P, R]=0$. But this implies $P[R, R]=0$. Since $P \neq 0$, $[R, R]$ must consist of zero divisors. As above, we have for $x \in [R, R]$, that $x^2 \in S$ and is a zero divisor, so $x^2=0$. If $x \in [R, R]$ and $x \neq 0$ then $(xr-rx)^2x=0$ and again we obtain a contradiction. Thus $[R, R]=0$ and R is commutative. Since we are assuming that R is not commutative, we must assume that R is prime.

By Theorem 4 every element of S' is regular in R. As $S' \subset Z$, it is a multiplicatively closed set, and so, we can consider the localization $R(S')^{-1}$, which is still a 2-torsion-free prime ring. Furthermore we can extend $*$ to $R(S')^{-1}$ by defining $(rs^{-1})^* = r^*s^{-1}$. A straightforward verification yields that this is indeed an involution extending $*$, with the property that rs^{-1} is symmetric exactly when r is.

If x is a non-zero-divisor in $R(S')^{-1}$ then xx^* is a unit, so x itself is a unit. Thus any proper ideal I of $R(S')^{-1}$ must consist of zero divisors. But as we have seen above this forces $I \subset Z$. Since $R(S')^{-1}$ is prime, either $I=0$ or $R(S')^{-1}$ is commutative. As we are assuming that R is not commutative we have that $R(S')^{-1}$ is simple. Now $S' \subset Z$ implies that if $x \in R(S')^{-1}$ then $x^2-(x+x^*)x+x^*x=0$, so $R(S')^{-1}$ is quadratic over its center. Hence

$R(S')^{-1}$ satisfies a polynomial identity of degree five over its center. Since $R(S')^{-1}$ is simple it must be either a division ring or the complete 2×2 matrix ring over a field by Kaplansky's Theorem. In the first case R is a domain and in the second we have (ii).

We now consider the second main case where R has no nonzero nilpotent elements. If R is prime, clearly it is a domain, so we obtain (i). We may henceforth assume that R is not prime. Let P be an ideal of R maximal with respect to exclusion of the multiplicative semigroup of regular elements of R. As above, P is prime and $P\neq 0$ since R is not prime. Further, P consists of zero divisors, so $P\cap S=0$. Suppose $xy=0$. Then $(yRx)^2=0$ and we have $yRx=0$. Since P is prime either $x\in P$ or $y\in P$. Suppose $x\notin P$. Since $xx^*=0$, by Theorem 4, we have $x^*\in P$. Thus if x is any zero divisor either $x\in P$ or $x^*\in P$. This says that $P\cup P^*$ is the set of zero divisors of R.

Suppose $x^2\in P$ but $x\notin P$. Now $x^2\in P$ means that x is a zero divisor and so $xx^*=x^*x=0$ by Theorem 4. Since $x\notin P$ we have $x^*\in P$. If $2x=s+k$ then $2x^*=s-k$. By Lemma 1, $s^2=k^2$ and $sk=ks$. Thus $(2x^*)^2=s^2+k^2-2sk\in P$ and $4x^2=s^2+k^2+2sk\in P$. We conclude that $2(s^2+k^2)=4s^2\in P\cap S=0$, and so, $s=0$. But $k^2=s^2=0$ and R has no nilpotent elements, so $x=0$. We have shown that R/P is a prime ring with no elements of square zero, so it is a domain.

If $x\in P\cap P^*$ then $x\in P^*$ implies $x^*\in P$. Hence $x+x^*$ and xx^* are in $P\cap S=0$. Thus $0=xx^*=-x^2$, so $x=0$. Hence R is a subdirect sum of R/P and R/P^* via $r\mapsto(r+P, r+P^*)$. Now R/P^* is naturally isomorphic to $(R/P)^{\mathrm{op}}$ via the map $r+P^*\mapsto r^*+P$, which is clearly a bijection and additive. Further $rt+P^*\mapsto(rt)^*+P=t^*r^*+P$ and $(r^*+P)\circ(t^*+P)=r^*\circ t^*+P=t^*r^*+P$. If $A=R/P$ we have that R is a subdirect sum of A and A^{op} via $r\mapsto(r+P, r^*+P)$. Clearly $*$ in R is induced by the involution in $A\oplus A^{\mathrm{op}}$ which interchanges co-ordinates.

COROLLARY 7. *If R is semiprime and S^-, the subring generated by S, is a domain, then R is either commutative, a domain, or $S\subset Z$ and $R(S')^{-1}$ is a complete 2×2 matrix ring over a field.*

PROOF. As in Theorem 6, if $S\subset Z$ then R is either commutative or $R(S')^{-1}$ is a complete 2×2 matrix ring over a field. Hence we may assume that $S\not\subset Z$, and so, that R has no nilpotent elements. If R is prime it must then be a domain. Suppose R is not prime and let P be an ideal maximal with respect to exclusion of regular elements. Then $P\neq 0$ and consists of zero divisors. Since $S\not\subset Z$ by Lemma 1.3 of [1] we have, as in Lemma 5, that $(2S)^-$ contains a nonzero ideal of $2R$. If I is the sum of all such ideals, then $I\subset(2S)^-$, $I^*=I$, and I consists only of regular elements of R. For if $x\in I-\{0\}$, then $x^*\in I$ so xx^* and x^*x are nonzero, $(2S)^-$ being a domain, and so regular in R by Theorem 4. But $PI\subset P\cap I=0$ since P consists of zero divisors. Thus $P=0$. This contradiction shows that R must be prime, and so, a domain.

Next we consider some examples to show that the various possibilities in Theorem 6 can occur.

Let $R=D[x,y]$ the polynomial ring in two indeterminates over any domain D of char$\neq 2$. Define * by interchanging x and y. For a noncommutative example one can take the quaternions over the integers with the usual conjugation as the involution. By considering the free algebra in at least two indeterminates over a field of char$\neq 2$, and defining * to reverse the order of the indeterminates in each monomial, we have a domain whose symmetric elements are not central.

An example for the second case of the theorem would be to let $R=D_2$, where D is any commutative domain of char$\neq 2$ and define * by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

For the third case let $R_1=F[x,y,z]$ and let J be the ideal generated by xy. Set $R=R_1/J$ and define * to interchange x and y, and fix z. A typical symmetric element of R is $f(z)+\sum_{i=0}^{n}(p_i(x)+p_i(y))z^i$ where p_i has no constant term. The ideal P in Theorem 6 can be taken to be (x), so $P^*=(y)$. Since $P+P^*\neq R$, R is indeed a subdirect sum as indicated and not a direct sum.

To get a noncommutative example for the third case let $F=(F[x,y]/(xy))\{w,t\}$ and define * to interchange x and y and reverse the order of each monomial in w and t. Again if $P=(x)$ and $P^*=(y)$ then R is a subdirect sum of $R/(x)$ and $R/(y)$ but $(x)+(y)\neq R$ implies that R is not a direct sum.

For a nonsemiprime example in any of the above cases, let R be one of the above examples and consider $R\{x,y,z\}$, the polynomial ring over R in 3 noncommuting indeterminates. Consider R to be a homomorphic image of the free algebra with identity over F in the indeterminates $\{t_i\}$. Let I be the ideal in $R\{x,y,z\}$ generated by t_ix, t_iy, and t_iz for all i and by all squares in $F\{x,y,z\}$ (without constants). Consider $A=R\{x,y,z\}/I$. Extend * to A by interchanging the order of indeterminates of monomials in x, y, and z and also sending x, y, z to their negatives. If N is the ideal generated by x, y, and z, then $N\subset K$, $N^3=0$ as in Theorem 3, and $A/N\cong R$.

Finally, we consider the situation in which R has an identity 1 and every element of S' is invertible. Again we will assume that S' is not empty.

THEOREM 8. *Suppose R is semiprime with* 1 *and that every element of S' is a unit in R. Then one of the following holds*:

(i) *$S\subset Z$ and R is the complete* 2×2 *matrix ring over a field*;

(ii) *R is a division ring*;

(iii) *$R\cong D\oplus D^{\mathrm{op}}$ where D is a division ring, D^{op} its opposite ring, and * in R is induced by interchanging co-ordinates in $D\oplus D^{\mathrm{op}}$.*

PROOF. Let us examine the possibilities which can occur by Theorem 6. First suppose R is a domain. If $x \in R$ and $x \neq 0$ then x^*x and $xx^* \in S'$, so x is a unit, and R is a division ring. Next assume (ii) of Theorem 6 holds. Since every element of S' is invertible, $R(S')^{-1} \cong R$, so R is the complete 2×2 matrix ring over a field. In the last case of Theorem 6, we need only show that $R \cong A \oplus A^{\mathrm{op}}$ and that A is a division ring. Note first that $(P+P^*) \cap S \neq 0$. Otherwise $P \subset K$ and since P consists of zero divisors in R, $x \in P$ implies $x^2 \in P \cap S = 0$. But if P is nil of index 2 it is a nilpotent ideal of R, a semiprime ring. Thus we must have $(P+P^*) \cap S \neq 0$. Since S' consists of units, $P+P^* = R$ and so $R \cong A \oplus A^{\mathrm{op}}$ by the Chinese Remainder Theorem. To show that R/P is a division ring it suffices to show that if $x \in R$, $x \neq 0$ is a zero divisor in R then $x \in P$ or $x+P$ is a unit in R/P. Suppose then, that $x \notin P$ and x is a zero divisor in R. By Theorem 4, $xx^* = x^*x = 0$ and so if $x = s+k$ we have by Lemma 1 that $s^2 = k^2$ and $sk = ks$ (note that we now have $\frac{1}{2} \in R$). Since R is a subring of the direct sum of two domains, R has no nilpotent elements. Hence if $s = 0$, then $k^2 = s^2 = 0$, so $x = 0$. Therefore $s \neq 0$ and we may consider $xs^{-1}/2 = \frac{1}{2}(1+s^{-1}k) = e$. Now $e^2 = e \neq 0$ and $e^* = 1-e$. Since e or e^* is in P we must have $e^* \in P$ if $x \notin P$. But this says that $(x+P)(s^{-1}/2+P) = 1+P$, so $x+P$ is a unit in R/P. Thus R/P is a division ring and the proof is complete.

COROLLARY 9 (OSBORN). *If R is a simple ring with 1 and every element of S' is invertible, then R is either a division ring or* $\dim_Z R \leqq 4$.

REFERENCES

1. I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, Ill., 1969.

2. N. Jacobson, *Structure theory for a class of Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 243–251. MR **33** #1337.

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