A PROPERTY OF \( y''' + p(x)y' + \frac{1}{2}p'(x)y = 0 \)

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Abstract. It is shown that if \( y''' + p(x)y' + kp'(x)y = 0 \) has an oscillatory solution, it has bases with 0, 1, 2, or 3 oscillatory elements.

It has recently been observed by Utz [3] that it is possible for a third order differential equation

\( y''' + p(x)y'' + q(x)y' + r(x)y = 0 \)

(1)

to have bases for its solution space consisting of 0, 1, 2, or 3 oscillatory elements, where to say a solution of (1) is oscillatory means it changes sign for arbitrarily large \( x \). The example he gave is

\( y''' - 3y'' + 4y' - 2y = 0, \)

(2)

which has solutions \( y_1 = e^x \sin x, \ y_2 = e^x \cos x, \ y_3 = e^x \). However, there is no loss in generality in assuming \( p(x) = 0 \) in (1) in investigating oscillating solutions, for \( p(x) \) can be eliminated from (1) by the transformation

\[ y = u \exp \left( -\frac{1}{3} \int p(t) \, dt \right). \]

Applying this transformation, (2) becomes

\( y''' = 0. \)

(3)

In an equation of the form

\( y''' + q(x)y' + r(x)y = 0 \)

(4)

where \( q(x) \) and \( r(x) \) are constant, it is not difficult to show that it is possible for (4) to have bases for its solution space consisting of 0, 1, 2, or 3 oscillating elements if and only if \( r = 0 \). But, this is equivalent, when \( q(x) \) and \( r(x) \) are constant, to saying that (4) is selfadjoint.
We now will prove the following generalization of the constant coefficient case for the general third order selfadjoint equation

$$y''' + b(x)y' + \frac{1}{b'}(x)y = 0,$$

where $b(x)$ and $b'(x)$ are assumed to be continuous on $(0, +\infty)$.

**Theorem.** If (5) has an oscillatory solution then its solution space has bases consisting of 0, 1, 2, or 3 oscillatory elements.

**Proof.** It is well known [1] that the general solution of (5) is

$$y(x) = k_1y_1(x) + k_2y_2(x) + k_3y_3(x),$$

where $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + \frac{1}{b}(x)y = 0,$$

If $u(x)$ and $v(x)$ are linearly independent solutions of (6), between two consecutive zeros of $u(x)$ there will be exactly one zero of $v(x)$ [2, p. 177]. Thus it is clear that $u(x)$ is oscillatory if and only if $v(x)$ is oscillatory, and that $u(x)$ and $v(x)$ cannot have a zero in common. From this it follows that if (6) is oscillatory, then $u(x)v(x)$ is an oscillatory solution of (5), where $u(x)$ and $v(x)$ are linearly independent solutions of (6). Using these facts, if (6) is oscillatory, it is easy to verify that if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (6), the following four bases of (5) have 3, 2, 1, and 0 oscillatory elements respectively:

$$y_1(x)y_2(x), \quad y_1(x)(y_1(x) + y_2(x)), \quad y_2(x)(y_1(x) + y_2(x));$$
$$y_1(x)y_2(x), \quad y_1(x)(y_1(x) + y_2(x)), \quad y_2(x)(y_1(x) + y_2(x));$$
$$y_1(x)y_2(x), \quad y_2(x)(y_1(x) + y_2(x)), \quad (y_1(x) + y_2(x))^2 + y_2(x).$$

It remains to show that if (5) has an oscillatory solution, then (6) is oscillatory. Suppose $y_1(x)$ and $y_2(x)$ are chosen so that

$$y_1(1) = 0 = y_2(1), \quad y_1'(1) = 1 = y_2(1).$$

Suppose $y_1(x)$ is not oscillatory. Then

$$y(x) = y_1(x) + k_1y_1(x)y_2(x) + k_2y_2(x)$$

is oscillatory for some choice of $k_1$ and $k_2$ not both zero. But a theorem of Birkhoff [1] says that if $u(x)$ and $v(x)$ are linearly independent solutions of (5) with at least one zero, then the zeros of $u(x)$ and $v(x)$ separate each other singly or in pairs. Thus since $y_1^2(1)=0$ it must have an infinity of zeros. Since $y_1(x)$ cannot have a double zero it must be oscillatory.
REFERENCES


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