HEREDITARY RADICALS IN JORDAN RINGS

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Abstract. The object of this paper is to examine some radical properties of quadratic Jordan algebras and to show that under certain conditions, \( R(\mathcal{B}) = \mathcal{B} \cap R(\mathcal{A}) \) where \( \mathcal{B} \) is an ideal of a quadratic Jordan algebra \( \mathcal{A} \), \( R(\mathcal{B}) \) is the radical of \( \mathcal{B} \), and \( R(\mathcal{A}) \) is the radical of \( \mathcal{A} \).

1. Preliminaries. We adopt the notation and terminology of an earlier paper [2] concerning quadratic Jordan algebras (defined by the quadratic operators \( U_x \)) as opposed to linear Jordan algebras (defined by the linear operators \( L_x \)). Thus we have a product \( U_{xy} \) linear in \( y \) and quadratic in \( x \) satisfying the following axioms as well as their linearizations:

- (UQJ I) \( U_1 = I \) (I the unit);
- (UQJ II) \( U_{U(x)y} = U_x U_y U_x \);
- (UQJ III) \( U_x V_y z = V_y U_x \) (\( V_x y z = (xyz) = U_x y z \)).

Throughout this paper \( \mathcal{A} \) will denote a quadratic Jordan algebra over an arbitrary ring of scalars \( \Phi \).

Define a property \( R \) of a class of rings (e.g. associative rings or Jordan rings) to be a radical property if it satisfies the following three conditions [1]:

(a) Every homomorphic image of an \( R \) ring is again an \( R \) ring.
(b) Every ring \( \mathcal{A} \) contains an \( R \) ideal \( R(\mathcal{A}) \) which contains every other \( R \) ideal of \( \mathcal{A} \). The maximal \( R \) ideal \( R(\mathcal{A}) \) is called the \( R \) radical of \( \mathcal{A} \).
(c) For \( \mathcal{B} \) an ideal of \( \mathcal{A} \), if \( \mathcal{B} \) and \( \mathcal{A} \mathcal{B} \) are \( R \) rings, then so is \( \mathcal{A} \).

An immediate consequence of this definition is \( R(\mathcal{A}/R(\mathcal{B})) = 0 \). If \( R(\mathcal{B}) = 0 \), \( \mathcal{B} \) is said to be \( R \) semisimple.

Many well-known radical properties, but not all, satisfy a further condition:

(d) Every ideal of an \( R \) ring is again an \( R \) ring (i.e. property \( R \) is inherited by ideals of an \( R \) ring).

If a radical property satisfies condition (d) that property is called hereditary.

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1 The results of this paper are included in the author's doctoral dissertation written at the University of Virginia.
In this paper we shall consider all radical properties \( R \) of \( \mathcal{J} \) such that if \( \mathcal{J} \) contains no \( R \) ideals then \( \mathcal{J} \) contains no absolute zero divisors (where \( z \in \mathcal{J} \) is an absolute zero divisor if \( U_z = 0 \)). Henceforth such radical properties will be called radical properties of type A.

Two of the more prominent radical properties are quasi-invertibility and nil: An element \( z \) belonging to a Jordan algebra \( \mathcal{J} \) is quasi-invertible (q.i.) with quasi-inverse \( w \in \mathcal{J} \) if \( 1 - z \) is invertible with inverse \( 1 - w \) in \( \Phi 1 \otimes \mathcal{J} \). A subset is called quasi-invertible if all of its elements are quasi-invertible in \( \mathcal{J} \). The maximal quasi-invertible ideal of an algebra is usually called the Jacobson radical and plays an important role in the structure theory. If we define an element \( z \in \mathcal{J} \) to be nilpotent if \( z^n = 0 \) for some \( n \) (here powers of an element are defined recursively by \( z^0 = 1, z^1 = z, z^{n+2} = Uzz^n \)), then a nil ideal is one all of whose elements are nilpotent. There is a unique maximal nil ideal containing all other nil ideals and this is called the nil radical. In [3] McCrimmon shows that nil and quasi-invertibility are radical properties of type A.

A lesser known radical property is that of antiprime: An algebra \( \mathcal{J} \) is strongly semiprime if it contains no absolute zero divisors. An algebra \( \mathcal{J} \) is an antiprime algebra (henceforth called a P-algebra) if no nonzero homomorphic image of \( \mathcal{J} \) is strongly semiprime (i.e. every homomorphic image of \( \mathcal{J} \) contains an absolute zero divisor).

**Proposition.** \( P \) is a radical property.

**Proof.** (a) Every homomorphic image \( \rho(\mathcal{J}) \) of a P-algebra \( \mathcal{J} \) is again a P-algebra for if not, then for some homomorphism \( \delta \), \( \delta(\rho(\mathcal{J})) = \rho(\mathcal{J}) \) is strongly semiprime and nonzero.

(b) For \( 23 \) an ideal of \( \mathcal{J} \) and for \( 23 \) a P-algebra, \( \mathcal{J} \) is also a P-algebra for if not, then some nonzero homomorphic image \( 3/R (R \in \mathcal{J}) \) is strongly semiprime.

**Case 1.** If \( B = R \), then \( 3/R \) is a strongly semiprime nonzero homomorphic image (projection) of \( 3/B \) which is assumed to be a P-algebra, a contradiction.

**Case 2.** If \( B \neq R \), examine \( 3/(3 \cap R) \) which is nonzero homomorphic image of the P-algebra \( B \). So there is an element \( x \in B \), \( x \notin R \), such that \( U_x B \subseteq R \). If \( U_x \mathcal{J} \subseteq R \), then \( x \notin R \) is a nonzero absolute zero divisor of \( 3/R \) which is assumed to be strongly semiprime: contradiction. If \( U_x \mathcal{J} \subseteq R \), examine \( U_{U_x j} \mathcal{J} \) (where \( U_x j \notin R \)). \( U_{U_x} \mathcal{J} = U_x U_x \mathcal{J} \subseteq U_x U_x B \subseteq U_x B \subseteq R \). So \( U_x j \) is a nonzero absolute zero divisor of \( 3/R \), a contradiction. Hence \( \mathcal{J} \) must be a P-algebra.

(c) Every P-algebra \( \mathcal{J} \) contains a P-ideal \( R \) which contains every other P-ideal of \( \mathcal{J} \). Let \( R = \sum \mathcal{B} \) be the sum of all the P-ideals of \( \mathcal{J} \). We claim
\( R \) is a \( P \)-ideal, for take any nonzero homomorphic image \( \rho(\mathfrak{R}) \) of \( \mathfrak{R} \):

\[
\rho(\mathfrak{R}) = \sum x_i \rho(\mathfrak{R}_x).
\]

We must find an absolute zero divisor in \( \rho(\mathfrak{R}) \). Since \( \rho(\mathfrak{R}) \neq 0 \), \( \rho(\mathfrak{R}_x) \neq 0 \) for some \( x \); and since \( \mathfrak{R}_x \) is a \( P \)-ideal, there exists a nonzero element \( z \in \rho(\mathfrak{R}_x) \) such that \( U_z \rho(\mathfrak{R}_x) = 0 \). If \( U_z \rho(\mathfrak{R}) = 0 \), \( z \) is an absolute zero divisor in \( \rho(\mathfrak{R}) \). So assume \( U_z \rho(\mathfrak{R}) \neq 0 \) and examine \( U_z \rho(\mathfrak{R})(x = U_y, y \in \rho(\mathfrak{R}), x \neq 0) \).

It is easy to see that \( P \) is a radical property of type A because the ideal spanned by all absolute zero divisors is clearly a \( P \)-ideal.

In the case of associative algebras one has the following theorem: For \( \mathfrak{R} \) an ideal of an associative algebra \( \mathfrak{A} \), \( R(\mathfrak{R}) = \mathfrak{R} \cap R(\mathfrak{A}) \) where \( R \) is the Jacobson radical. In [4] McCrimmon proves this theorem for quadratic Jordan algebras. We now prove this theorem for all hereditary radical properties of type A.

2. The proof.

**Lemma.** For \( \mathfrak{B} \) an ideal of a quadratic Jordan algebra \( \mathfrak{J} \), for any radical property \( R \) of type A, and for \( x \in \mathfrak{B} \), \( U_x \mathfrak{B} \subseteq R(\mathfrak{B}) \Rightarrow x \in R(\mathfrak{B}) \).

**Proof.** \( U_x \mathfrak{B} \subseteq R(\mathfrak{B}) \) implies \( U_x(\mathfrak{B}/R(\mathfrak{B})) = 0 \), i.e., \( x \) is an absolute zero divisor of \( \mathfrak{B}/R(\mathfrak{B}) \). But since \( R \) is a radical property of type A, \( R(\mathfrak{B}/R(\mathfrak{B})) = 0 \) implies \( R(\mathfrak{B}) \) contains no nonzero absolute zero divisors. Hence \( x = 0 \) or \( x \in R(\mathfrak{B}) \).

**Theorem 1.** For \( \mathfrak{B} \) an ideal of a quadratic Jordan algebra \( \mathfrak{J} \), and for \( R \) any radical property of type A, \( R(\mathfrak{B}) \subseteq \mathfrak{B} \cap R(\mathfrak{J}) \).

**Proof.** Since \( R(\mathfrak{B}) \subseteq \mathfrak{B} \), it is sufficient to show that \( R(\mathfrak{B}) \) is an \( R \) ideal of \( \mathfrak{J} \) and therefore \( R(\mathfrak{B}) \subseteq R(\mathfrak{J}) \). That is, we must show

1. \( U_{R(\mathfrak{B})} \mathfrak{J} \subseteq R(\mathfrak{B}) \),
2. \( U_{R(\mathfrak{B})} \mathfrak{B} \subseteq R(\mathfrak{B}) \).

**Note.** We may as well assume that \( \mathfrak{J} \) is unital since any quadratic Jordan algebra \( \mathfrak{J} \) can be imbedded in a unital quadratic Jordan algebra \( \mathfrak{J}' = \Phi \subseteq \mathfrak{J} \), and any ideal in \( \mathfrak{J} \) will be an ideal in \( \mathfrak{J}' \). To prove (1), let \( z \in U_{R(\mathfrak{B})} \mathfrak{J} \), i.e., let \( z \) be a finite linear combination of elements of the form \( U_{x_i} y_i \) where \( x_i \in R(\mathfrak{B}) \) and \( y_i \in \mathfrak{J} \). In view of our lemma, since \( z \in \mathfrak{B} \) it is sufficient to prove \( U_z \mathfrak{B} \subseteq R(\mathfrak{B}) \). Also since \( U_z = U_{s_U}(x_i y_i) = \sum U_{x_i y_i} + \sum_{i,j} U_{x_i} y_i, U_{x_j} y_j \), it is now clear that we will be done if \( U_{U_{x_j} y_j} \mathfrak{B} \subseteq R(\mathfrak{B}) \) for \( x \in R(\mathfrak{B}) \) and \( y \in \mathfrak{J} \); for then \( U_{x_j y_j} \mathfrak{B} \subseteq R(\mathfrak{B}) \) implies \( U_{x_j} y_j \in R(\mathfrak{B}) \) which in turn implies \( U_{U_{x_j} y_j} \mathfrak{B} \subseteq R(\mathfrak{B}) \). But \( U_{U_{x_j} y_j} \mathfrak{B} = U_{x_j} U_{x_j} \mathfrak{B} \subseteq U_{x_j} R(\mathfrak{B}) \) (since \( R(\mathfrak{B}) \) is an ideal of \( \mathfrak{B} \)) \( U_{x_j} \mathfrak{B} \subseteq R(\mathfrak{B}) \).

To prove (2) we now let \( z \in U_{R(\mathfrak{B})} \mathfrak{J} \), i.e., let \( z \) be a finite linear combination of elements of the form \( U_{x_i} y_i \) where \( x_i \in \mathfrak{J} \) and \( y_i \in R(\mathfrak{B}) \). Again
we are done if $U_{U(a)}\mathfrak{B} \subseteq R(\mathfrak{B})$ for $x \in \mathfrak{J}$ and $y \in R(\mathfrak{B})$. By repeated application of the lemma it is sufficient to show $U_{n} \mathfrak{B} \subseteq R(\mathfrak{B})$ for $p=U_{b}, q=U_{b}, r=U_{y}, s=U_{x}$, where $x \in \mathfrak{J}, y \in R(\mathfrak{B})$, $b$ and $b'$ are arbitrary elements of $\mathfrak{B}$. But

$$
U_{x} \mathfrak{B} = U_{U(a)} b = U_{U(a)} b U_{U(a)} = U_{U(a)} b U_{U(a)} U_{U(a)} = (U_{x} U_{b} U_{x})(U_{x} U_{b} U_{x}) U_{U(a)} \mathfrak{B}
$$

So

$$
U_{x} \mathfrak{B} \subseteq (U_{x} U_{b} U_{x})(U_{x} U_{b} U_{x}) U_{U(a)} \mathfrak{B} \quad \text{(since $\mathfrak{B}$ is an ideal of $\mathfrak{J}$)}
$$

$$
\subseteq (U_{x} U_{b} U_{x})(U_{x} U_{b} U_{x}) U_{y} \mathfrak{B} \quad \text{(since $y \in R(\mathfrak{B})$, an ideal of $\mathfrak{B}$)}
$$

$$
\subseteq (U_{x} U_{b} U_{x})(U_{x} U_{b} U_{x}) R(\mathfrak{B}) \quad \text{(by a linearization of UQJ III)}
$$

$$
\subseteq U_{x} R(\mathfrak{B}) \subseteq R(\mathfrak{B}).
$$

**Theorem 2.** If $R$ is any hereditary radical property of type A in a quadratic Jordan algebra $\mathfrak{J}$ and if $\mathfrak{B}$ is an ideal of $\mathfrak{J}$, then $R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J})$.

**Proof.** By Theorem 1 it is sufficient to show $R(\mathfrak{B}) \supseteq \mathfrak{B} \cap R(\mathfrak{J})$. But since $R$ is a hereditary radical property, any ideal of an $R$ ring is again an $R$ ring. In particular, $\mathfrak{B} \cap R(\mathfrak{J})$ is an ideal of $R(\mathfrak{J})$ and is therefore an $R$ ring and an ideal in $\mathfrak{B}$. Therefore $\mathfrak{B} \cap R(\mathfrak{J}) \subseteq R(\mathfrak{B})$.

**Corollary.** For $R$ nil or quasi-invertible, and for $\mathfrak{B}$ an ideal of $\mathfrak{J}$, $R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J})$. So, $\mathfrak{J}$ $R$-semisimple $\Rightarrow$ $\mathfrak{B}$ $R$-semisimple.

**Proof.** It has been mentioned that nil and quasi-invertibility are radical properties of type A. It remains to show that they are hereditary radical properties. But it is clear that every ideal of a nil ring is again nil. Furthermore, the explicit expression for the quasi-inverse of an element $z$ is $w = U_{(1-z)}^{-1}(z^2-z)$ since

$$
(1 - z)^{-1} = U_{(1-z)}^{-1}(1 - z) = U_{(1-z)}^{-1}(1 - z)^2 - U_{(1-z)}^{-1}(z^2 - z) = 1 - w.
$$

If $\mathfrak{B}$ is any ideal then the quasi-inverse of any quasi-invertible element of $\mathfrak{B}$ also belongs to $\mathfrak{B}$. Quasi-invertibility, therefore, is a hereditary radical property.
References


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