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THE CONJUGACY PROBLEM FOR THE GROUP OF ANY TAME ALTERNATING KNOT IS SOLVABLE[1]

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ABSTRACT. The theorem of the title is proved by using the techniques of Weinbaum along with the prime decomposition of knots and an extension of some small cancellation techniques of Lyndon and Schupp.

The argument to prove the theorem of the title is based on the proof of Weinbaum [6] for prime alternating knots which in turn employs the methods of Lyndon [2] and Schupp [5]. To save repetition, familiarity with Weinbaum's paper is assumed. Knot, in this paper, will always mean tame knot in general position with respect to a projection plane (i.e. all multiple points of the projection are isolated double points of which there are finitely many).

If K is a knot then K has a projection due to Schubert [4] which may be described as follows. Start with a circle, which we will call the Schubert circle, with some k disjoint small open arcs deleted. Around the exterior of the circle place disjoint projections of k prime knots each with a small open arc (not including a double point) deleted. Join the endpoints of the arcs on the prime projections pairwise to the endpoints of the open arcs on the circle without introducing further double points.

We also want to deal with knot projections having parts arranged around a fixed circle. We need a condition weaker than primeness on the parts.

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[1] This proof is simpler than a proof announced by Appel [1] which was based on a Wirtinger presentation, and therefore that proof will not be published. The authors are indebted to Professor Wolfgang Haken for many helpful suggestions.

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DEFINITION. A knot projection is called *elementary* if (in Weinbaum's terminology):

(1) any vertex is on the boundary of four distinct domains;

(2) any two domains have at most one edge in common.

(Our definition of "elementary" is exactly the same as Weinbaum's definition of "common".)

Being elementary is a property of projections while being prime is a property of knots independent of particular projections.

If K is a prime knot, then any projection of K having a minimal number of crossings is elementary. Composite knots may have elementary projections. (We conjecture that if a projection is both alternating and elementary then it is a projection of a prime knot.)

We define below a "standard projection" of a knot. The standard projection will consist of elementary parts arranged around a fixed circle. Based on Schubert's ideas, we give an algorithm which, starting from any projection of a knot K, produces a standard projection of K. For the purposes of the present paper there are two advantages of replacing prime by elementary. First, given any alternating projection, the standard projection produced is also alternating. Second, the given procedure is clearly effective.

DEFINITION. Let π be a projection of a knot K. Let Γ be a simple closed curve which intersects π in precisely two points and such that there are vertices of π both interior to and exterior to Γ. Then Γ is called a *separating curve* for π.

DEFINITION. Let π be a projection of a knot K. We call π standard if

(i) any vertex of π is on the boundary of four distinct domains;

(ii) there exists a bounded domain X_1 of π having an edge in common with the unbounded domain X_0 such that any separating curve for π lies entirely in X_0 and X_1.

In a standard projection, condition (2) in the definition of elementary is violated only by the domains X_0 and X_1.

Given any projection π of a knot K we proceed as follows to obtain a standard projection of K. Note that π cannot have any vertex lying on fewer than three distinct domains since K is a knot. Now, if π contains a vertex v such that only three distinct domains meet at v, then a simple closed curve Δ may be drawn intersecting π only in v and separating π into two parts. By appropriately turning over the part of the projection inside Δ, the crossing v can be eliminated. (See Figure 1.) (This untwisting is operation $\Omega.4$ of Reidemeister [3] and is discussed fully there.) If π is alternating then so is the new projection. Since this operation is effective and reduces the number of crossings, in a finite number of steps we reach a projection having four distinct domains meeting at each vertex.

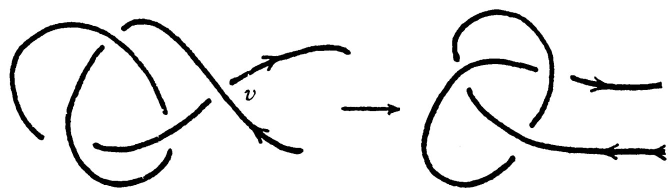

FIGURE 1

Now suppose that (ii) of the definition of standard fails. Choose any bounded domain X_1 which has an edge e in common with the unbounded domain X_0. If Γ is a separating curve lying in any pair of domains X_a and X_b other than X_0 and X_1, then Γ is called a *bad separating curve*. If Γ is a bad separating curve, then any vertex v on an edge of π intersected by Γ is called a bad vertex. Clearly v lies on X_a and X_b.

Let Γ be a bad separating curve. It may be that Γ contains X_1 in its interior. Then, since X_1 is adjacent to X_0, Γ must lie in X_0 and some X_a. Hence there is another bad separating curve Γ' lying in X_0 and X_a which contains the part of π exterior to Γ. If Γ does contain X_1 in its interior we replace Γ by Γ' and change notation. We thus may assume that Γ is a bad separating curve which does not contain X_1 in its interior.

The following is an intuitive description of what is essentially Schubert's argument. Take a sphere S enclosing that part of the knot whose projection is enclosed by Γ. The knot pierces S exactly twice. Shrink S and its interior to a "suitably small" size. Regard the point at which the knot enters S as being fixed. By stretching the knot, pull S along the knot until reaching an "unused" portion of the original edge e. (See Figure 2.)

In view of the previous discussion we define a *transfer* operation on the projection π as follows. Let Γ be a bad separating curve not containing X_1 in its interior. Let π' be the part of π interior to Γ and let π'' be the part of π exterior to Γ. Obtain π^* from π as follows.

(1) Delete π' and replace it by an arc interior to Γ joining the ends of π''.

(2) Delete a small arc δ from the original edge e. Replace δ by a curve Σ (geometrically) similar to π' but small enough so that Σ does not intersect π except at the joining points and such that orientation in going around the knot is preserved.

(3) Since the projection enters Γ once and leaves Γ once, each vertex interior to Γ is traversed twice, once as an undercrossing and once as an overcrossing. Hence, if π is alternating then π' is alternating and similarly π'' is alternating. When Σ is moved to the edge e, there may be a pair of consecutive undercrossings (or overcrossings) consisting of the last vertex traversed in Σ and the next vertex traversed. If this happens, turn

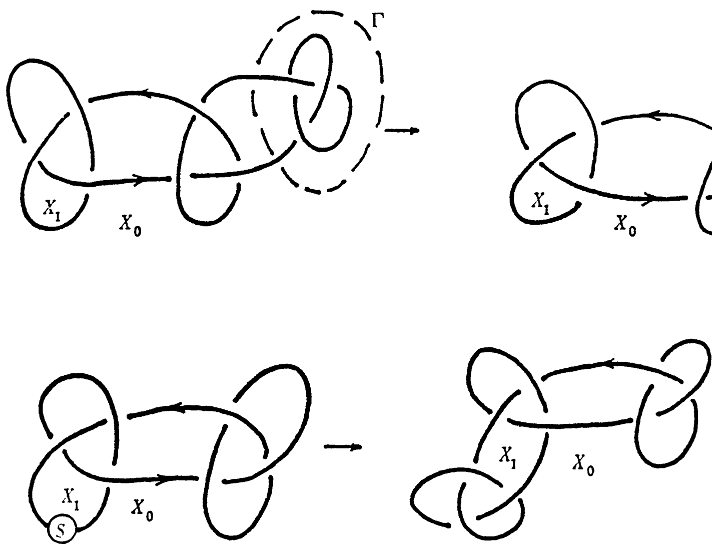

FIGURE 2

over Σ. This reverses overcrossings and undercrossings in Σ and the projection π^* is now alternating.

Now a transfer operation does not change the number of vertices but does reduce the number of bad vertices. Hence, after a finite number of transfers we obtain a standard projection. The fact that the knot type is not changed is justified by the intuitive description preceding the definition in which we perform allowable deformations on the knot.

DEFINITION. Let π be a projection which is standard but not elementary. Remove a single (interior) point from each edge e_i common to the boundaries of X_0 and X_1. This divides π into parts J_i which, if their loose ends are joined, are elementary projections. We call the J_i the *elementary parts* of π.

Let us recall what Weinbaum proves. He uses the Dehn presentation of a knot group. (We will use the same letters to denote both the domains of a projection and the corresponding generators of the group.) Weinbaum shows that, for any projection of a knot, if the relator X_0 is deleted the resulting group, G, is the free product of the knot group and an infinite cycle. We will work with the same group G and speak of the Dehn presentation of G. Weinbaum used the hypothesis of primeness only to ensure that the projection is elementary. He proved that if the projection is elementary and alternating then the Dehn presentation of G satisfies the small cancellation conditions C(4) and T(4). (Weinbaum denotes the triangle condition by T_3 while we use T(4).) The hypothesis that the projection is elementary and alternating is used only to verify C(4). Although not stated explicitly in Weinbaum, the proof supplied shows that for any projection of a knot, the Dehn presentation of G satisfies T(4).

Let R be a symmetrized set of relators. A sequence $u_1r_1u_1^{-1}, \cdots, u_nr_nu_n^{-1}$ of conjugates of elements of R is called *minimal* if the product $w=u_1r_1u_1^{-1}\cdots u_nr_nu_n^{-1}$ is not a product of fewer conjugates of elements of R. In constructing diagrams to study word and conjugacy problems one need only consider diagrams of minimal sequences. (See Schupp [5].) In the present context we consider finite symmetrized sets R of defining relators each of which has length four. We say that *R satisfies* C(4) *and* T(4) *for minimal sequences if*

(1) If two elements r_1, r_2 of R cancel two or more letters then either $r_1r_2=1$ or r_1r_2 is already an element of R (C(4)).

(2) If r_1, r_2, r_3 are elements of R with cancellation in all the products r_1r_2, r_2r_3 and r_3r_1, then $r_1r_2r_3$ is a product of two or fewer elements of R (T(4)).

It is easy to see that if R satisfies C(4) and T(4) for minimal sequences then a diagram of a minimal sequence is a (4, 4) map. Thus such a group has solvable word and conjugacy problems exactly as if the relators satisfied C(4) and T(4) in the absolute sense. We will show that if K is an alternating knot then the group defined by Weinbaum can be presented so that C(4) and T(4) are satisfied for minimal sequences.

Let K be any alternating knot and let π_1 be any alternating projection of K. If π_1 is not standard apply the procedure of the first part of the paper to obtain a standard alternating projection π of K. We assume that π is standard but not elementary (otherwise Weinbaum's argument is immediately applicable).

In each elementary part J_i there are two crossings on X_0 and X_1. Since π is alternating, without loss of generality we may assume that the one of these which is the first vertex traversed in the parametrization of π is traversed by an overcrossing curve. This vertex is called the overcrossing of J_i. Similarly the other is called the undercrossing of J_i. We define domains U_i, V_i, P_i, Q_i as follows: U_i and V_i are the domains other than X_0 and X_1 on the undercrossing of J_i with U_i adjacent to X_0 and V_i adjacent to X_1. P_i and Q_i are the domains other than X_0 and X_1 on the overcrossing of J_i with P_i adjacent to X_0 and Q_i adjacent to X_1. (See Figure 3.) It is possible that $P_i=U_i$ or $Q_i=V_i$ but if both equations held then either P_i and Q_i are adjacent along two edges, contradicting J_i being elementary, or K consists of two linked curves; hence not both can hold.

We have the following results from Weinbaum.

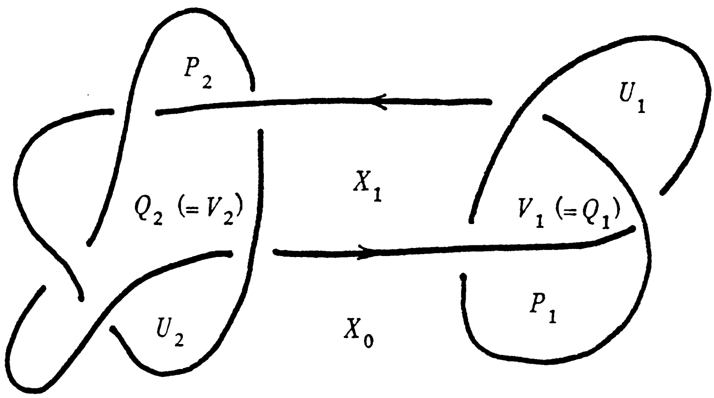

FIGURE 3

(1) The relators arising from each elementary part satisfy C(4) among themselves.

(2) The relators as a whole satisfy T(4).

We note that independent of the orientation of the knot, in the Dehn presentation for G, the symmetrized set determined by the relator corresponding to the overcrossing on J_i is determined by $X_1X_0^{-1}P_iQ_i^{-1}$, while that corresponding to the undercrossing is determined by $X_0^{-1}X_1V_i^{-1}U_i$. The presentation we consider is the Dehn presentation augmented by the symmetrized set determined by relators $P_iQ_i^{-1}Q_jP_j^{-1}$ (which we call an *overcrossing relator*) and $V_i^{-1}U_iU_j^{-1}V_j$ (called an *undercrossing relator*) with i, j ranging over all pairs of distinct indices of elementary parts of K. We call these additional relators the *derived relators*.

Two observations seem to give some idea of why this proof succeeds while extension of this method to nonalterating knots may be impossible. First we note that if two relators from distinct elementary parts have any generators in common these must be among X_0 and X_1. Second, for any derived overcrossing relator, in reading two successive occurrences of generators from the same elementary part, the exponents are positive followed by negative. In reading successive occurrences of generators from different elementary parts the exponents are negative then positive. For the derived undercrossing relators the exponent patterns are opposite. To verify C(4) we must show that if r_1 and r_2 are relators such that r_1r_2 has length four or less then r_1r_2 is either trivial or another relator. From Weinbaum's work we may assume that r_1 is a derived relator. The argument is symmetric for over and undercrossings so we will assume that r_1 is in the symmetrized set determined by $P_iQ_i^{-1}Q_jP_j^{-1}$.

Case 1. r_1 *is* $P_iQ_i^{-1}Q_jP_j^{-1}$. Now if r_2 is a relator corresponding to a crossing on J_j then it must be $P_jQ_j^{-1}X_1X_0^{-1}$ by C(4) on relators from elementary parts, and then $r_1r_2=P_iQ_i^{-1}X_1X_0^{-1}$. If r_2 is a derived overcrossing relator it must be $P_jQ_j^{-1}Q_kP_k^{-1}$ and r_1r_2 is $P_iQ_i^{-1}Q_kP_k^{-1}$. Now suppose r_2 is determined by a derived undercrossing relator. Then it must begin $U_jV_j^{-1}$ where $U_j=P_j$, $V_j=Q_j^{-1}$, but this exponent pattern is impossible for an undercrossing relator.

Case 2. r_1 *is* $Q_i^{-1}Q_jP_j^{-1}P_i$. Here clearly r_2 must be a derived relator. If it is determined by an overcrossing relator it must be r_1^{-1}. If it is determined by an undercrossing relator it must begin $U_i^{-1}U_j$ where $U_i=P_i$, $U_j=P_j$. But again, no such relator exists because of the exponent pattern.

To verify T(4) we may assume that r_1 is a derived relator and that r_1, r_2, r_3 are such that each of r_1r_2, r_2r_3 and r_3r_1 have precisely one cancelling letter.

Case 1. r_1 *is* $P_iQ_i^{-1}Q_jP_j^{-1}$. If r_2 is an overcrossing relator then the product r_1r_2 is not minimal. Thus this case may be ignored.

Subcase (i). $P_j=U_j$ and $r_2=U_jU_k^{-1}V_kV_j^{-1}$. Then r_3 has the form $V_jABP_i^{-1}$. Since $P_j=U_j$, we must have $V_j\neq Q_j$. Hence, r_3 is the derived undercrossing relator $V_jV_i^{-1}U_iU_j^{-1}$. But then $P_i=U_j$, contradicting the uniqueness of the elementary part in which P_i lies.

Subcase (ii). The situation where r_3 is a derived relator is the same as that where r_1 and r_2 are derived relators. Thus, we may now suppose that neither r_2 nor r_3 are derived relators. Then r_2 comes from the elementary part J_j and r_3 comes from J_i. Thus the generator cancelled in the product r_2r_3 is either X_0 or X_1. Now r_2 must begin with P_j. The knot projection is a graph all of whose vertices have degree four. By a standard chromatic lemma of graph theory, its domains may be "properly" two-colored. Thus if D and E are opposite domains at one vertex they cannot be adjacent at another vertex. Since P_j is adjacent to X_0 at one vertex the last letter in r_2 must be X_0^{-1}. By C(4) in the elementary part J_j, $r_2=P_jQ_j^{-1}X_1X_0^{-1}$. Hence, $r_3=X_0ABP_i^{-1}$. By C(4) in J_i, $r_3=X_0X_1^{-1}Q_iP_i^{-1}$. Thus r_2r_3 is not minimal.

Case 2. r_1 *is* $Q_i^{-1}Q_jP_j^{-1}P_i$. As before, if r_2 is a derived overcrossing relator then the product r_1r_2 is not minimal.

Subcase (i). If r_2 is a derived undercrossing relator then $P_i=U_i$ and $r_2=U_i^{-1}V_iV_k^{-1}U_k$. Then r_3 is $U_k^{-1}ABQ_i$. Since $Q_i\neq V_i$, $r_3=P_i^{-1}P_jQ_j^{-1}Q_j$, yielding the contradiction $P_i=U_k$ where $i\neq k$.

Subcase (ii). Suppose that neither r_2 nor r_3 is derived. Then both come from the elementary part J_i. If r_1, r_2, r_3 all cancel, then so do r_1', r_2, r_3 where $r_1'=Q_i^{-1}X_1^{-1}X_0^{-1}P_i$. This violates T(4) unless one of r_2 or r_3 is $(r_1')^{-1}$. This assumption yields a contradiction since it implies that either r_2 or r_3 is not cyclically reduced. For example, if $r_2=(r_1')^{-1}$, then r_3 must begin with Q_i^{-1} and end with Q_i for cancellation to occur in r_2r_3 and r_3r_1. This completes the proof of the theorem.

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