PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

A NOTE ON POLYHEDRAL BANACH SPACES

ALAN GLEIT AND ROBERT McGUIGAN

ABSTRACT. We give a sufficient condition for an infinite-dimensional Banach space X to be polyhedral. If X^* is an L-space this condition is also necessary.

1. **Introduction.** In this paper all Banach spaces will be over the scalar field of real numbers. If X is a Banach space, the unit ball of X is denoted $S(X)$. If K is a convex subset of a vector space, $\mathrm{Ext}(K)$ is the set of all extreme points of K. Thus $\mathrm{Ext}\, S(X)$ is the set of all extreme points of the unit ball of X. A Banach space X is polyhedral iff for every finite-dimensional subspace $E \subseteq X$, $S(E)$ has only finitely many extreme points.

The following lemma is essentially contained in [2]. We include a brief proof for the sake of completeness.

LEMMA 1.1. *If E is a finite-dimensional Banach space such that $S(E)$ has infinitely many extreme points, then $S(E^*)$ has infinitely many extreme points.*

PROOF. Recall that a point $x \in E$ with $\|x\|=1$ is a smooth point of $S(E)$ iff there is a unique functional $f \in E^*$ such that $\langle f, x\rangle = \|f\| = 1$. According to a theorem of Mazur [8], the set of smooth points of $S(E)$ is dense in the boundary of $S(E)$. Let $\mathcal{S} \subseteq \mathrm{Ext}\, S(E^*)$ be the set of all functionals of norm one in E^* that attain their norms at smooth points of $S(E)$. We have

$$S(E) = \bigcap_{f \in \mathcal{S}} \{x \mid \langle f, x\rangle \leqq 1\}.$$

If $\mathcal{S}$ were finite, then $S(E)$ would be the intersection of a finite number of half-spaces and hence a polyhedron. This contradiction implies that $\mathcal{S}$ is infinite.

REMARK. Klee also shows in [2] that a finite-dimensional Banach space E is polyhedral iff E^* is polyhedral. Lindenstrauss has shown [6] that no infinite-dimensional conjugate space can be polyhedral.

THEOREM 1.2. *For an infinite-dimensional Banach space X consider the*

Received by the editors July 25, 1971.

AMS 1970 *subject classifications*. Primary 46B99; Secondary 46E99, 52A05.

Key words and phrases. Polyhedral Banach space, Lindenstrauss space, biface, L-space.

following three properties:

1. *There do not exist a weak*-accumulation point q of* Ext $S(X^*)$ *and* $x \in X$ *satisfying* $\langle q, x\rangle = \|x\| = \|q\| = 1$.

2. X *is polyhedral.*

3. *No subspace of* X *is linearly isometric to c.*

Then (1)$\Rightarrow$(2)$\Rightarrow$(3). *When* X^* *is an L-space, then* (1) *through* (3) *are equivalent statements.*

PROOF. We establish the contrapositives of the required implications. (1)$\Rightarrow$(2). Suppose X is not polyhedral. Then there is a finite-dimensional subspace $E \subseteq X$ such that $S(E)$ has infinitely many extreme points. By the previous lemma, $S(E^*)$ has infinitely many extreme points. Since $S(E^*)$ is norm-compact, we can choose a sequence (f_n) of distinct elements of Ext $S(E^*)$ such that $f_n \to g \in E^*$. Now g attains its norm on $S(E)$ (since E is reflexive) and $\|g\| = 1$.

If $f \in \text{Ext } S(E^*)$ then $\{\hat{f} \in S(X^*) \mid \|\hat{f}\| = \|f\| = 1 \text{ and } \hat{f}|E = f\}$ is a weak*-closed face of $S(X^*)$ and hence contains an element of Ext $S(X^*)$. Thus, for each n there is some $\hat{f}_n \in \text{Ext } S(X^*)$ such that $\hat{f}_n|E = f_n$. By weak*-compactness of $S(X^*)$, $(\hat{f}_n)$ has a weak*-cluster point $h \in S(X^*)$. Clearly $h|E = g$. Thus $\|h\| = 1$ and h attains its norm.

(2)$\Rightarrow$(3). Suppose c is linearly isometric to a subspace of X. Then X is not polyhedral since c is not polyhedral.

The proof of (3)$\Rightarrow$(1) for spaces whose duals are L-spaces requires some additional lemmas, and is therefore postponed until the next section.

REMARK. Implication (1)$\Rightarrow$(2) of the previous theorem generalizes Klee's theorem [3] that c_0 is polyhedral, since clearly c_0 satisfies (1).

2. **Lindenstrauss spaces.** We first recall some terminology. A vector lattice V is an L-space if its norm satisfies

$$\|p + q\| = \|p\| + \|q\|, \quad p, q \in V^+, \quad \text{and} \quad \|p\| = \|p^+\| + \|p^-\|.$$

A Banach space whose dual space is an L-space is called a Lindenstrauss space (see [**1**] for terminology). If a Lindenstrauss space can be ordered in a way compatible with the duality and the natural order in the L-space, it is called a simplex space.

Suppose that W is a normed space. We say that $r \in W$ dominates p, written $p \prec r$, if $r = p + q$ and $\|r\| = \|p\| + \|q\|$. A nonempty subset H of $S(W)$ is a biface if:

B_1. H is convex and symmetric.

B_2. If $x \neq 0$ is in H then $x/\|x\|$ is in H.

B_3. If $q \in H$ and $p \prec q$, then $p \in H$.

It is easily seen that $\text{Ext}(H) \subseteq \text{Ext } S(W)$ [**1**, Lemma 4.5].

LEMMA 2.1. *Let W be a normed space and $F \subseteq S(W)$ a convex subset consisting of elements of norm one. If $H = \mathrm{co}(F \cup -F)$ is a biface, then F is a face.*

PROOF. Let $x, y \in S(W)$ and $0 < \lambda < 1$ satisfy $f = \lambda x + (1-\lambda)y \in F$. Since $1 = \|f\| \leqq \lambda \|x\| + (1-\lambda)\|y\| \leqq \lambda + (1-\lambda) = 1$, we conclude that $\|x\| = \|y\| = 1$, and that f dominates both λx and $(1-\lambda)y$. From properties B_2 and B_3 we easily see that $x, y \in H$. Thus there are elements $p_x, p_y, q_x, q_y \in F$ and $0 \leqq \alpha, \beta \leqq 1$ satisfying

$$x = \alpha p_x - (1-\alpha)q_x, \qquad y = \beta p_y - (1-\beta)q_y.$$

Then

$$f = (\lambda\alpha p_x + (1-\lambda)\beta p_y) - (\lambda(1-\alpha)q_x + (1-\lambda)(1-\beta)q_y).$$

Let $\mu = \lambda\alpha + (1-\lambda)\beta$. Then

$$1 - \mu = \lambda(1-\alpha) + (1-\lambda)(1-\beta).$$

Assume $0 < \mu < 1$. Then

$$a = \mu^{-1}(\lambda\alpha p_x + (1-\lambda)\beta p_y) \in F,$$
$$b = (1-\mu)^{-1}(\lambda(1-\alpha)q_x + (1-\lambda)(1-\beta)q_y) \in F.$$

But then

$$\mu f + (1-\mu)b = \mu^2 a + (1-\mu)^2 b \in F.$$

Since every element of F has norm one, we get

$$1 = \|\mu^2 a + (1-\mu)^2 b\| \leqq \mu^2 + (1-\mu)^2$$

which implies $\mu = 0$ or $\mu = 1$. This is a contradiction. Hence $\mu = 0$ or $\mu = 1$. If $\mu = 0$, then $\alpha = \beta = 0$ and so $f = -q_x - q_y \in F \cap -F$ which is clearly impossible. So $\mu = 1$, and thus $\alpha = \beta = 1$. Hence $x = p_x \in F$ and $y = p_y \in F$.

The following result is due to A. Lazar.

PROPOSITION 2.2. *Let H be a nonzero biface in an L-space V. Then there is a face $F \subseteq S(V)$ such that $H = \mathrm{co}(F \cup -F)$.*

PROOF. Let $F = \{p \in H \mid \|p\| = 1, p = p^+\}$. F is convex since V is an L-space. Let $q \in H$, $q \neq 0$. Then $q = q^+ - q^-$. If $q^+ \neq 0$ we have $q^+/\|q^+\| \in F$. Otherwise $q^-/\|q^-\| \in F$, and in any case, $F \neq \varnothing$. Hence $0 \in \mathrm{co}(F \cup -F)$.

Since H is convex and symmetric, $\mathrm{co}(F \cup -F) \subseteq H$. Conversely, let $q \in H$. Then $q = q^+ - q^-$, and $\|q\| = \|q^+\| + \|q^-\|$. Hence $q > q^+$, $q > q^-$, and, by B_3, $q^+, q^- \in H$. Thus $q^+/\|q^+\|$ and $-q^-/\|q^-\|$ belong to F. The equality

$$\frac{q}{\|q\|} = \frac{\|q^+\|}{\|q\|}\left(\frac{q^+}{\|q^+\|}\right) - \frac{\|q^-\|}{\|q\|}\left(\frac{q^-}{\|q^-\|}\right)$$

together with $0\in\text{co}(F\cup -F)$ clearly yields $q\in\text{co}(F\cup -F)$. Hence $H=\text{co}(F\cup -F)$. Now Lemma 2.1 implies that F is a face and the proof is complete.

For completeness we quote here, in our terminology, the generalization by Lazar and Lindenstrauss of the Edwards separation theorem.

THEOREM 2.3 [5, THEOREM 2.1]. *Let X be a Lindenstrauss space. Let $g:S(X^*)\to(-\infty,\infty]$ be a concave weak*-lower semicontinuous function satisfying*

$$g(v^*)+g(-v^*)\geqq 0,\qquad v^*\in S(X^*).$$

Let H be a weak-closed biface of $S(X^*)$. Assume that f is a weak*-continuous affine symmetric real-valued function on H such that $f\leqq g|H$. Then there is an element $v\in X$ which satisfies*

1. $\langle v^*, v\rangle=f(v^*)$, $v^*\in H$,
2. $\langle y^*, v\rangle\leqq g(y^*)$, $y^*\in S(X^*)$.

COROLLARY 2.4. *Let X be a Lindenstrauss space and H a weak*-closed biface of $S(X^*)$. Let f be a weak*-continuous affine symmetric real-valued function on H. Then there is an element $v\in X$ which satisfies*

1. $\langle v^*, v\rangle=f(v^*)$, $v^*\in H$,
2. $\|v\|=\|f\|_\infty$.

PROOF. We may take $g\equiv\|f\|_\infty$ in Theorem 2.3.

LEMMA 2.5. *Let V be an L-space. Let $P=\{p_\alpha|\alpha\in A\}$ be a subset of* Ext $S(V^*)$ *such that $p_\alpha\neq\pm p_\beta$ for all $\alpha,\beta\in A$.* Let $\{p_n\}$ *be a sequence drawn from P. Then if $\sum|\alpha_n|\leqq 1$,*

$$\sum_{n=1}^{\infty}\alpha_n p_n=0\Rightarrow\alpha_n=0,\qquad \textit{each } n.$$

In particular, P is linearly independent.

PROOF. By changing the sign of α_n where necessary we may assume $p_n=p_n^+$ for all n. The result then follows from obvious triangle inequalities.

We say that a subset D of $S(X^*)$, X a Lindenstrauss space, is *symmetrically dilated* if for each $p\in D$, the extreme points of some weak*-closed biface containing p are contained in D.

COROLLARY 2.6. *Let X be a Lindenstrauss space. Suppose $\{p_n\}\subseteq$* Ext $S(X^*)$ *converges weak* to $q\in\text{co}(x_1,\cdots,x_N)$ for $x_i\in$* Ext $S(X^*)$, *$i=1,\cdots,N$. Suppose $p_n\neq\pm p_m$, $\pm x_i$ and $x_i\neq\pm x_j$ for all n,m,i,j. Then*

$$F=\overline{\text{co}}\left(\bigcup\{p_n\},x_1,\cdots,x_N\right)$$

is a proper infinite-dimensional weak-closed face of $S(X^*)$.*

PROOF. Let $K=\mathrm{co}(\pm x_1, \cdots, \pm x_N)$. K is clearly the smallest weak*-closed biface containing q [**1**, Proposition 4.6]. Let $D=\{\pm p_n\}\cup K$. Then D is symmetrically dilated and compact. Hence the closed convex hull H of D is a biface of $S(X^*)$ [**1**, Theorem 5.8]. Clearly

$$H = \overline{\mathrm{co}}\left(\bigcup \{\pm p_n\}, \pm x_1, \cdots, \pm x_N\right) = \mathrm{co}(F \cup -F).$$

Also,

$$F = \left\{\sum \alpha_n p_n + \sum \beta_i x_i \mid \alpha_n \geqq 0, \beta_i \geqq 0, \sum \alpha_n + \sum \beta_i = 1\right\},$$

and so

$$H = \left\{\sum \alpha_n p_n + \sum \beta_i x_i \mid \sum |\alpha_n| + \sum |\beta_i| \leqq 1\right\}.$$

On H we define a weak*-continuous affine symmetric function f by

$$f\left(\sum \alpha_n p_n + \sum \beta_i x_i\right) = \sum \alpha_n + \sum \beta_i.$$

f is well defined by Lemma 2.5. Clearly $f|F\equiv 1$. By Corollary 2.4, there is a $v\in X$ which extends f and satisfies $\|v\|=\|f\|_\infty=1$. As each $q\in F$ satisfies $\langle q, v\rangle=1$ we have that $\|q\|=1$ for each $q\in F$. Hence, by Lemma 2.1, F is a face of $S(X^*)$. F is infinite dimensional since

$$P = \left\{\bigcup \{p_n\}, x_1, \cdots, x_N\right\}$$

is linearly independent by Lemma 2.5.

We are now prepared to complete the proof of Theorem 1.2. First we recall that if X is a Lindenstrauss space, Lazar [**4**, Theorem 3] has shown that X has no subspace linearly isometric to c iff $S(X^*)$ has no proper infinite-dimensional weak*-closed face.

PROOF OF THEOREM 1.2 (CONTINUED). We shall first show that the negation of (1) implies the existence of a proper infinite-dimensional weak*-closed face of $S(X^*)$ if X is a separable Lindenstrauss space. Assuming (1) is false, we can find a weak*-accumulation point q of Ext $S(X^*)$ and a $v\in X$ satisfying $\langle q, v\rangle=\|v\|=\|q\|=1$. Let

$$G = \{p \in S(X^*) \mid p(v) = 1\}.$$

Then G is a weak*-closed proper nonempty face of $S(X^*)$. If it is infinite dimensional, we are done. So assume that G is finite dimensional. By Corollary 2.5 we easily conclude that $G=\mathrm{co}(x_1, \cdots, x_N)$ for some $x_i\in\mathrm{Ext}\, S(X^*)$. Since $S(X^*)$ is weak*-metrizable, there is a sequence $\{p_n\}\subseteq \mathrm{Ext}\, S(X^*)$ weak*-converging to q. Without loss of generality, we may assume that $p_n\neq\pm p_m$, x_i for all n, m, i. But then Corollary 2.6 provides us with a proper infinite-dimensional weak*-closed face of $S(X^*)$.

We shall now reduce the nonseparable case to the separable case by showing that if (1) is false for a nonseparable Lindenstrauss space X then it is false for a separable Lindenstrauss subspace of X. Let X be nonseparable and let q be a weak*-accumulation point of $\operatorname{Ext} S(X^*)$ with $x_0 \in X$ such that

$$\langle q, x_0 \rangle = \|x_0\| = \|q\| = 1.$$

Choose $q_1 \in \operatorname{Ext} S(X^*)$ such that $|1-\langle q_1, x_0\rangle|<2^{-1}$. Let $x_1 \in X$ satisfy $\langle q_1, x_1\rangle = \|x_1\| = 1$. Such an x_1 exists by Corollary 2.4. Define

$$F_1 = \{x^* \in S(X^*): \langle x^*, x_1\rangle = 1\}.$$

Now choose $q_2 \in \operatorname{Ext} S(X^*)$ such that $|1-\langle q_2, x_0\rangle|<2^{-2}$ and $q_2 \notin F_1$. By Lazar's theorem [4, Theorem 3], F_1 is finite dimensional. By Corollary 2.4 choose $x_2 \in X$ such that x_2 vanishes on F_1 and $\langle q_2, x_2\rangle = \|x_2\| = 1$. Inductively we construct two sequences $\{q_n\} \subset \operatorname{Ext} S(X^*)$, $\{x_n\} \subset X$ such that

1. $|1-\langle q_n, x_0\rangle|<2^{-n}$,
2. $q_n \notin F_i = \{x^* \in S(X^*): \langle x^*, x_i\rangle = 1\}$ for $i<n$,
3. $\|x_n\| = \langle q_n, x_n\rangle = 1$,
4. x_n vanishes on F_i for $i<n$.

Let Z be a separable Lindenstrauss subspace of X which contains the sequence $\{x_n | n=0, 1, 2, \cdots\}$ [7, Theorem 4.4 (a) and Theorem 6.1, (2)⇔(12)]. Let $\varphi: X^*/Z^\perp = Z^*$ be the canonical map. For $n \geqq 1$,

$$P_n = \{z^* \in S(Z^*): \langle z^*, x_n\rangle = 1\}$$

is a weak*-closed face of $S(Z^*)$ and $\varphi(q_n) \in P_n$. There is a $z_n^* \in \operatorname{Ext} P_n \subset \operatorname{Ext} S(Z^*)$ such that $1 \geqq \langle z_n^*, x_0\rangle \geqq \langle \varphi(q_n), x_0\rangle > 1-2^{-n}$. Now if $n<m$ we have $\langle z_n^*, x_m\rangle = 0$ since, by the Hahn-Banach theorem, $\varphi^{-1}(z_n^*) \cap F_n \neq \varnothing$. Recalling that $\langle z_n^*, x_n\rangle = 1$, we note that $z_n^* \neq z_m^*$ for $n \neq m$. Now if z^* is a weak*-limit point of $\{z_n^*\}$ we have $\langle z^*, x_0\rangle = 1$.

COROLLARY 2.7. *Let X be a simplex space. Then the following statements are equivalent*:

1. *There is no weak*-accumulation point q of* $\operatorname{Ext} S(X^*)$ *with* $\|q\|=1$.
2. *X is polyhedral.*
3. *No subspace of X is linearly isometric to c.*

PROOF. We need only show that for each weak*-accumulation point q of $\operatorname{Ext} S(X^*)$ satisfying $\|q\|=1$, there is a $v \in X$ satisfying $\langle q, v\rangle = 1 = \|v\|$. There is no loss of generality in assuming $q \in X^{*+}$ since $\operatorname{Ext} S(X^*) \subset X^{*+} \cup X^{*-}$, and X^{*+} and X^{*-} are closed. Let F be the minimal weak*-closed face of $K = S(X^*) \cap X^+$ containing q. Clearly F consists of elements all of norm one. On the face $[0, 1]\, F$ define a weak*-continuous function by $h(\alpha f)=\alpha$. As K is a simplex, h is well defined. Apply Edwards separation theorem to get an isometric extension $v\in X$ of the function h. Then $\langle q, v\rangle=1=\|v\|$.

ACKNOWLEDGMENT. We would like to thank the referee for his many valuable comments. In particular, the extension of Theorem 1.2 to the nonseparable case is due to him.

REFERENCES

1. E. Effros, *On a class of real Banach spaces*, Israel J. Math. **9** (1971), 430–458.

2. V. Klee, *Some characterizations of convex polyhedra*, Acta Math. **102** (1959), 79–107. MR **21** #4390.

3. ———, *Polyhedral sections of convex bodies*, Acta Math. **103** (1960), 243–267. MR **25** #2512.

4. A. J. Lazar, *Polyhedral Banach spaces and extensions of compact operators*, Israel J. Math. **7** (1969), 357–364. MR **41** #796.

5. A. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. **126** (1971), 165–194.

6. J. Lindenstrauss, *Notes on Klee's paper "Polyhedral sections of convex bodies,"* Israel J. Math. **4** (1966), 235–242. MR **35** #703.

7. ———, *Extension of compact operators*, Mem. Amer. Math. Soc. No. 48 (1964). MR **31** #3828.

8. S. Mazur, *Über konvexe Mengen in linearen normierten Raumen*, Studia Math. **4** (1933), 70–84.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002