PURE STATES WITH THE RESTRICTION PROPERTY

BRUCE A. BARNES

Abstract. Conditions are given which imply that a pure state of a $B^*$-algebra $A$ restricts to a pure state of some maximal commutative $*$-subalgebra of $A$.

1. Introduction. A pure state $\rho$ of a $B^*$-algebra $A$ has the restriction property if there exists a maximal commutative $*$-subalgebra $C$ of $A$ such that the restriction of $\rho$ to $C$ is a pure state of $C$ (i.e. $\rho$ is a nonzero multiplicative linear functional on $C$). The work of R. Kadison and I. Singer in [4] raises the question of whether or not each pure state of a $B^*$-algebra has the restriction property. This question was answered by J. Aarnes and R. Kadison for a special class of $B^*$-algebras $A$. They prove that when $A$ is separable and has an identity, then each pure state of $A$ has the restriction property [1, Theorem 2]. Again in the case when $A$ is separable, C. Akemann in [2] removed the requirement that $A$ have an identity and made other improvements in the result of Aarnes and Kadison (including a proof that in this case a pure state $\rho$ of $A$ is the unique extension of a pure state of some maximal commutative $*$-subalgebra of $A$). However, the general question remains open.

In this note we give several new conditions on a pure state $\rho$ of a $B^*$-algebra which imply that $\rho$ has the restriction property. $A$ is a $B^*$-algebra throughout. Let $a \mapsto \pi(a)$ be a $*$-representation of $A$ on a Hilbert space $H$. A positive functional $\rho$ is represented by $\pi$ if there is $\xi \in H$, $\|\xi\| = 1$, such that $\rho(a) = (\pi(a)\xi, \xi)$ for all $a \in A$. A pure state of $A$ is always represented by some irreducible $*$-representation of $A$; see [3, pp. 32, 33, 37] for details.

Now let $\rho$ be a pure state of $A$ which is represented by an irreducible $*$-representation $\pi$ of $A$ on a Hilbert space $H$. We prove that if either $H$ is separable or $\pi(A)$ contains $F(3t)$, the algebra of bounded operators on $H$ with finite dimensional range, then $\rho$ has the restriction property. The proofs of these results are indebted to the ideas of Aarnes and Kadison in [1].

2. The results. Let be $H$ a Hilbert space. $B(H)$ is the algebra of bounded operators on $H$. When $H$ is a subspace of $H'$ and $B$ is a...
nonempty subset of $\mathcal{B}(\mathcal{H})$, then $B\mathcal{H}$ is the linear span of the vectors $\{T\psi | T \in B, \psi \in \mathcal{H}\}$. $[B\mathcal{H}]$ is the closure of $B\mathcal{H}$ in $\mathcal{H}$. When $T \in \mathcal{B}(\mathcal{H})$, then $\mathcal{N}(T)$ is the null space of $T$ and $\mathcal{R}(T)$ is the range of $T$.

**Lemma.** Let $\mathcal{H}$ be a separable Hilbert space and assume that $A$ is a closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ such that $[A\mathcal{H}]=\mathcal{H}$. Then there exists $T \in A$, $T \geq 0$, such that $\mathcal{N}(T)=0$.

**Proof.** Since $[A\mathcal{H}]=\mathcal{H}$, then given any $\psi \in \mathcal{H}$, $\psi \neq 0$, there exists $S \in A$ such that $S\psi \neq 0$. Therefore for each $\psi \in \mathcal{H}$, $\psi \neq 0$, we can choose $T \in A$ such that $T\psi \geq 0$ and $T\psi(\psi) \neq 0$. Let $U_\psi = \{\xi \in \mathcal{H} | T_\psi(\xi) \neq 0\}$. The collection $\{U_\psi | \psi \in \mathcal{H}, \psi \neq 0\}$ is an open cover for $\mathcal{H}\{0\}$. By the open cover of a separable metric space is Lindelöf (every open cover has a countable subcover). It follows that there exists a sequence $(T_n) \subset A$ such that $T_n \geq 0$ and $\bigcap_{n=1}^{+\infty} T_n(\mathcal{H}) = 0$. Let $a_n = (2\|T_n\|)^{-1}$, and set $T = \sum_{n=1}^{+\infty} a_n T_n$. If $T\psi = 0$, then $\sum_{n=1}^{+\infty} a_n (T_n\psi, \psi) = 0$. Therefore $T_n\psi, \psi = 0$ for each $n$. But then $T_n\psi = 0$, which implies $T\psi = 0$ for each $n$. Therefore $\psi = 0$.

When $D$ is a nonempty subset of $A$, we let

$$\mathcal{C}(D) = \{a \in A | ad = da \text{ for all } d \in D\}.$$  

If $D$ is selfadjoint, then $\mathcal{C}(D)$ is a closed $*$-subalgebra of $A$.

**Theorem 1.** Let $a \rightarrow \tau(a)$ be an irreducible $*$-representation of $A$ on a separable Hilbert space $\mathcal{H}$. If $\rho$ is a positive functional represented by $\tau$, then $\rho$ has the restriction property.

**Proof.** There exists $\xi \in \mathcal{H}$, $\|\xi\|=1$, such that $\rho(a) = \langle \tau(a)\xi, \xi \rangle$ for all $a \in A$. Let $K = \{a \in A | \rho(a*a) = 0\} = \{a \in A | \tau(a)\xi = 0\}$. Set $A_0 = K \cap K^*$ and $\mathcal{H}_0 = \{\xi\}$. Given $a \in A_0$ and $\psi \in \mathcal{H}$, we have $\langle \tau(a)\psi, \xi \rangle = \langle \psi, \tau(a^*)\xi \rangle = 0$. Therefore $\tau(A_0)\mathcal{H}_0 \subset \mathcal{H}_0$. Let $E$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$. Then

$$E\tau(a) = \tau(a) \quad \text{for all } a \in A_0.$$  

By [3, Corollaire (2.8.4)], $\tau(A)$ acts strictly irreducibly on $\mathcal{H}$. Therefore there exists $v \in A$ such that $\tau(v)\xi = \xi$. Since $\tau(v + v^* - v^*v)\xi = \xi$, we may assume that $v = v^*$. Set $u = 2v - v^2$. Then $I - \tau(u) = (I - \tau(v))^2 \geq 0$ where $I$ is the identity operator on $\mathcal{H}$. If $\psi \in \mathcal{H}$, $\langle (I - \tau(v))\psi, \xi \rangle = \langle \psi, (I - \tau(v))^2 \xi \rangle = 0$. Therefore

$$E(I - \tau(u)) = I - \tau(u).$$  

Given $\psi \in \mathcal{H}_0$, the transitivity theorem [3, Théorème (2.8.3)] implies that there exists $a \in A$ such that $a = a^*$, $\tau(a)\xi = 0$, and $\tau(a)\psi = \psi$. Then $a \in A_0$, and this proves that $\tau(A_0)\mathcal{H}_0 = \mathcal{H}_0$. By the Lemma there exists $w \in A_0$, $w \geq 0$,
such that $\mathcal{N}(\pi(w)) \cap \mathcal{H}_0 = 0$. Set $S = I - \pi(u) + \pi(w)$. Since $I - \pi(u) \geq 0$, then $\mathcal{N}(S) \cap \mathcal{H}_0 = 0$. Let $y = w - w$, and choose $C_0$ a maximal commutative $*$-subalgebra of $\mathcal{C}(y) \cap A_0$. Let $C$ be the closed commutative $*$-subalgebra of $A$ generated by $y$ and $C_0$. We prove that $C$ is a maximal commutative $*$-subalgebra of $A$. Assume that $b = b^*$ and $b \in \mathcal{C}(C)$. Let $b_0 = b - \rho(b)y$. Then $b_0 = b^*$ and $b_0 \in \mathcal{C}(C)$. Using (1) and (2) we have $ES = E(I - \pi(u) + \pi(w)) = I - \pi(u) + \pi(w) = S$. Also $\pi(b_0)S = S\pi(b_0)$. Then

$$(E\pi(b_0) - \pi(b_0)E)S = S\pi(b_0) - \pi(b_0)S = 0.$$  

Since $\mathcal{N}(S) \cap \mathcal{H}_0 = 0$ and $S = S^*$, then $(\mathcal{H}(S))^- = \mathcal{H}_0 = \mathcal{H}(E)$. Therefore $(E\pi(b_0) - \pi(b_0)E)E = 0$. It follows that $\pi(b_0)E = E\pi(b_0)$. Then there exists a scalar $\lambda$ such that $\pi(b_0)\xi = \lambda \xi$. Note that $\rho(y) = \rho(u - w) = \rho(2v - v^2) = (\pi(2v - v^2)\xi, \xi) = 1$. Therefore $\lambda = (\pi(b_0)\xi, \xi) = \rho(b_0) = \rho(b - \rho(b)y) = 0$. Then $b_0 \in \mathcal{C}(y) \cap A_0$, and it follows that $b_0 \in C_0$. But then $b \in C$. This proves that $C$ is a maximal commutative $*$-subalgebra of $A$.

$p$ is nonzero on $C$ since $p(y) = 1$. It remains to be shown that $p$ is multiplicative on $C$. Given $a \in C_0$, then $ya \in \mathcal{C}(y)$. Also $\pi(\pi(a)\xi) = \pi(p(a)\xi) = 0$ and similarly $\pi(a^*y)\xi = 0$. Therefore $ya$ and $(ya)^*$ are in $\mathcal{C}(y) \cap A_0$. Thus $ya \in C_0$. Furthermore $\pi(y)\xi = (\pi(u - \pi(w))\xi = \pi(u)\xi = \xi$. Thus $\pi(y^n - y)\xi = 0$ for any positive integer $n$. Then $y^n - y \in \mathcal{C}(y) \cap A_0$, and therefore $y^n - y \in C_0$ for each positive integer $n$. It follows that every element of $C$ has the form $\lambda y + a$ for some scalar $\lambda$ and some $a \in C_0$. Then given $\lambda, \mu$ scalars and $a, b \in C_0$,

$$\rho((\lambda y + a)(\mu y + b)) = \lambda \mu = \rho(\lambda y + a)\rho(\mu y + b).$$

This completes the proof of the theorem.

In the case where $A$ has an identity, the proof of Theorem 1 can be considerably simplified.

**Theorem 2.** Let $a \to \pi(a)$ be a $*$-representation of $A$ on a Hilbert space $\mathcal{K}$ with the property that $\mathcal{F}(\mathcal{K}) = \pi(A)$. If $p$ is a positive functional represented by $\pi$, then $p$ has the restriction property.

**Proof.** Assume that $\rho(a) = (\pi(a)\xi, \xi)$ for all $a \in A$, where $\xi \in \mathcal{K}$, $\|\xi\| = 1$. Let $K = \{a \in A | \pi(a)\xi = 0\}$, and set $A_0 = K \cap K^*$. Let $E$ be the orthogonal projection with one dimensional range containing $\xi$. By hypothesis there exists $e \in A$, $e = e^*$, such that $\pi(e) = E$. Choose $C_0$ a maximal commutative $*$-subalgebra of $\mathcal{C}(e) \cap A_0$. Let $C$ be the closed commutative $*$-subalgebra of $A$ generated by $e$ and $C_0$. Assume that $b = b^* \in \mathcal{C}(C)$. Set $b_0 = b - \rho(b)e$. Note that $\rho(e) = (E\xi, \xi) = 1$, so that $\rho(b_0) = 0$. Then $\pi(b_0)E = E\pi(b_0)$. Therefore there exists a scalar $\lambda$ such that $\pi(b_0)\xi = \lambda \xi$. Then $\lambda = (\pi(b_0)\xi, \xi) = \rho(b_0) = 0$. It follows that $b_0 \in \mathcal{C}(e) \cap A_0$, so that by the definition of $C_0$, $b_0 \in C_0$. Then $b \in C$. This proves that $C$ is
a maximal commutative *-subalgebra of $A$. The proof that $\rho$ is a nonzero multiplicative functional on $C$ proceeds as in the last paragraph of the proof of Theorem 1 with $e$ in place of $y$.

When $A$ is a GCR algebra (postliminaire) and $a \mapsto \pi(a)$ is an irreducible *-representation of $A$ on a Hilbert space $\mathcal{H}$, then it is well known that $\pi(A) \subseteq \pi(A)$; see [3, Théorème (4.3.7)]. Therefore we have as a corollary of Theorem 2:

**Corollary.** A pure state of a GCR algebra $A$ has the restriction property.

**References**


Department of Mathematics, University of Oregon, Eugene, Oregon 97403