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PL INVOLUTIONS OF SOME 3-MANIFOLDS[1]

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ABSTRACT. Let h_1 and h_2 be PL involutions of connected, oriented, closed, irreducible 3-manifolds M_1 and M_2, respectively. Let a_i, $i=1, 2$, be a fixed point of h_i such that near a_i the fixed point sets of h_i are of the same dimension. Then we obtain a PL involution $h_1 \# h_2$ on $M_1 \# M_2$ induced by h_i by taking the connected sum of M_1 and M_2 along neighborhoods of a_i. In this paper, we study the possibility for a PL involution h on $M_1 \# M_2$ having a 2-dimensional fixed point set F_0 to be of the form $h_1 \# h_2$, where M_i are lens spaces. It is shown that: (1) if F_0 is orientable, then $M_1 = -M_2$ and h is the obvious involution, (2) if the fixed point set F contains a projective plane, then $M_1 = M_2 =$ a projective 3-space, and in this case, F is the disjoint union of two projective planes and h is unique up to PL equivalences, (3) if F contains a Klein bottle K, then F is the disjoint union of a Klein bottle and two points.

1. **Introduction.** Let M be a closed, oriented 3-manifold which is the connected sum $M_1 \# M_2$ of two irreducible 3-manifolds M_1 and M_2, and let h be a PL involution of M with a fixed point set F containing a nonorientable surface F_0. Since F_0 is one-sided, it would seem that h cannot interchange M_1-part and M_2-part and h must be obtained from involutions h_i of M_i, $i=1, 2$, by attaching two involutions along an invariant neighborhood of fixed points a_i of h_i, where near a_i the fixed point sets are of the same dimension.

Kwun [4] proved that no lens spaces except the real projective space P_3 admits orientation reversing PL involutions, and in case of P_3 there exists a unique PL involution up to PL equivalences and F is the disjoint union of a projective plane and a point. Hence it leads us to consider the possibility of the above question when M_i are isomorphic to lens spaces.

2. **PL involutions of lens spaces.** Let h_1 and h_2 be PL involutions of M_1 and M_2, respectively, where M_i, $i=1, 2$, is isomorphic to a lens space (not necessarily having the natural orientation). The connected sum $M=$

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$M_1 \# M_2$ is obtained by removing the interior of a nice 3-cell from each, and then matching the resulting boundaries using an orientation reversing homeomorphism.

If Z_2 acts on M, $H^*(M/Z_2; Q) \approx H^*(M; Q)^{Z_2}$, and since the involution reverses orientation, we obtain $H^3(M; Q)^{Z_2}=0$, and hence $\chi(M/Z_2)=1$. Therefore $2\chi(M/Z_2)=\chi(F)+\chi(M)$ implies $\chi(F)=2$. Let $b_i(F)$ denote the ith mod 2 betti number of F. Then $\sum b_i(F) \leqq \sum b_i(M) \leqq 1+2+2+1=6$ (see [2, p. 42]). This, together with $\chi(F)=2$ and dim F is even, implies that the possible fixed point sets are S^2, the disjoint union of two projective planes, the disjoint union of a projective plane and a point, the disjoint union of a Klein bottle and two points, the disjoint union of a Klein bottle and S^2, the disjoint union of $S^1 \times S^1$ and S^2, and the disjoint union of $S^1 \times S^1$ and two points.

Now we first consider the case that F contains an orientable surface.

THEOREM 1. *Let h be a PL involution of $M=M_1 \# M_2$. If the fixed point set F contains an orientable surface, then F is a 2-sphere and $M_2=-M_1$, h being the obvious involution.*

PROOF. Let S be an orientable surface in F. By the Alexander duality theorem [9], over the rationals Q, F separates M into two parts U and V. Hence $M=2\bar{U}$, and we have $U \xrightarrow{i} M \xrightarrow{r} U$, such that ri is the identity, where i is the inclusion and r is a retraction. Hence we obtain the exact sequence

$$H_i(\bar{U}; Q) \xrightarrow{i_*} H_i(M; Q) \xrightarrow{r_*} H_i(\bar{U}; Q)$$

where $r_* i_*$ is the identity. Hence $H_i(U; Q)=0$ for $i=1, 2$, and therefore F must be a 2-sphere.

By Milnor [8], we may say that U is isomorphic to M_1-part and V is isomorphic to M_2-part, and since U and V must be interchanged by h, $M_2=-M_1$ and h is the obvious involution. This proves the theorem.

Hence we have eliminated the case that F is the disjoint union of $S^1 \times S^1$ and S^2, the disjoint union of $S^1 \times S^1$ and two points, or the disjoint union of a Klein bottle and S^2.

Since the case where a 2-dimensional component of F is orientable has been taken care of, we have to consider only the case where each 2-dimensional component of F is nonorientable.

THEOREM 2. *Let M be $M_1 \# M_2$, where M_i is isomorphic to a lens space, and h a PL involution on M. If a real projective plane P_2 is contained in the fixed point set F of h, then $M=P_3 \# P_3$.*

PROOF. Suppose that h fixes a real projective plane P and assume that M has been triangulated so that h is simplicial and the simplicial neighborhood U of P is an invariant regular neighborhood of P. Moreover,

assume that $h|_{U-P}$ is fixed point free. Let $p: N_1 \to M$ be the double covering obtained from M by cutting along P. Since U is orientable, but P is not, (U, P) is homeomorphic to (N, P), where N is the mapping cylinder of a double covering $S^2 \to P$. Let U' and $(M-U)'$ be the connected manifolds obtained from U and $(M-U)$ by attaching a 3-cell to each. Then by Milnor [8], U' is isomorphic to S^3, M_1, or M_2. But since $\pi_1(U')$ is Z_2, U' cannot be isomorphic to S^3, and we may say U' is isomorphic to M_1. Now $h|_U$ can be extended to an orientation reversing PL involution h' of U' defined by the cone over $h|_{\mathrm{Bd}(U)}$. Then by Kwun [4], $U' \approx M_1 \approx P_3$. Similarly $M_2 \approx P_3$. This completes the proof.

THEOREM 3. *Let $M = M_1 \# M_2$ and h a PL involution of M. If a projective plane is contained in the fixed point set F of h, then F is the disjoint union of two projective planes, and h being of the form $h_1 \# h_2$ is unique.*

PROOF. Suppose that a projective plane P is contained in F and assume that h is simplicial, the simplicial neighborhood U of P is invariant regular, and $h|_{U-P}$ is fixed point free. We have seen in Theorem 2 that $h|_{M-U}$ can be extended to an orientation reversing PL involution h' of P_3. Let F' be the fixed point set of h'. Then by Kwun [3], F' is the disjoint union of a projective plane and a point p. By the way we extended h to h', p is the cone vertex. Therefore $h|_{M-U}$ has a projective plane as the fixed point set F'. Hence F is the disjoint union of two projective planes.

Let P and P' be the two projective planes whose union is F. Take U and U' to be disjoint invariant regular neighborhoods of P and P', respectively. Then by Milnor [8], $M-(U \cup U')$ is isomorphic to $S^2 \times [0, 1]$, and by Livesay [7], there is a unique involution on $S^2 \times [0, 1]$ up to PL equivalences. Hence it suffices to analyse $h|_U$ and $h|_{U'}$. But by Kwun [3] these are unique, and $h = h_1 \# h_2$, where h_1 and h_2 are the extensions of $h|_U$ and $h|_{U'}$ to P_3. This proves the theorem.

From now on we consider the case that F contains a Klein bottle K. The only possible F is the disjoint union of K and two points. The union of K and two points can be the fixed point of an involution h. For, let the disjoint union of a projective plane A_i and a point p_i, $i=1, 2$, be the fixed point set of h_i of P_3. Taking the connected sum $P_3 \# P_3$ along an invariant neighborhood of a_i, $a_i \in A_i$, we obtain a PL involution $h_1 \# h_2$ whose fixed point set is the disjoint union of a Klein bottle and two points. Hence we have

THEOREM 4. *Let $M = M_1 \# M_2$ and h a PL involution of M with a Klein bottle in the fixed point set F. Then F is the disjoint union of a Klein bottle and two points.*

The uniqueness question for h in case F is the disjoint union of a Klein bottle and two points is not settled, but the following theorem gives some idea how the Klein bottle is located in $P_3 \# P_3$.

THEOREM 5. *Let M be the connected sum $P_3 \# P_3$ and let K be the Klein bottle in the fixed point set F of an involution h of M. Then $\pi_1(M-U)$ is the integers Z and $M-U$ is homeomorphic to $D^2 \times S^1$.*

For the proof of Theorem 5, we first prove the following lemma.

LEMMA 6. *Let h be a PL involution of $M=P_3 \# P_3$. Then there exists a PL involution $h': S^1 \times S^2 \to S^1 \times S^2$ such that $p'h'=hp'$, where $p': S^1 \times S^2 \to P_3 \# P_3$ is a 4-to-1 covering projection.*

PROOF. Consider the usual 2-to-1 covering map p: $S^1 \times S^2 \to M$. Let $H=p_\# \pi_1(S^1 \times S^2)$ and $G=\pi_1(M)$. Suppose $h_\# H \neq H$. Since $[G:H]=[G:h_\# H]=2$, there is no inclusion relation between H and $h_\# H$. Let $L=H \cap h_\# H$. Then L is a normal subgroup of G, and

$$H/L=H \cdot h_\# H/h_\# H=G/h_\# H=Z_2$$

implies $[H:L]=2$. Hence $[G:L]=4$. Furthermore $h_\# L=L$. Hence by the lifting theorem there is a PL involution h' on $S^1 \times S^2$ such that $p'h'=hp'$, where p' is 4-to-1. If $H=h_\# H$, then the construction is similar and easier. This proves the lemma.

PROOF OF THEOREM 5. Consider the double covering $p: N \to M$ obtained from M cutting along K. Then $p^{-1}(K)$ is homeomorphic to $S^1 \times S^1$. Since each component of $M_1 - p^{-1}(K)$ maps homeomorphically onto $M-K$, $p^{-1}(U)$ is a collar of $p^{-1}(K)$. So $\mathrm{Bd}(U)$ is homeomorphic to $S^1 \times S^1$. By the Mayer-Vietoris sequence

$$H_2(M) \to H_1(S^1 \times S^1) \to H_1(\bar{U}) + H_1(M - U) \to H_1(M) \to 0$$

we obtain that $H_1(M - U)$ is a group of rank 1.

Now we show that $\pi_1(M-U)$ is the integers Z. Let $p: S^1 \times S^2 \to P_3 \# P_3$ be a 4-to-1 covering such that $h_\# p_\# \pi_1(S^1 \times S^2)=p_\# \pi_1(S^1 \times S^2)$. Then $h': S^1 \times S^2 \to S^1 \times S^2$ exists such that the following diagram commutes:

$$\begin{array}{ccc} S^1 \times S^2, \tilde{x}_0 & \xrightarrow{h'} & S^1 \times S^2, \tilde{x}_0 \\ \downarrow p & & \downarrow p \\ P_3 \# P_3, x_0 & \xrightarrow{h} & P_3 \# P_3, x_0 \end{array} \qquad (x_0 \in K).$$

Consider the component A of $\tilde{x}_0$ in $p^{-1}(K)$. Then A is either $S^1 \times S^1$ or a Klein bottle K.

Case (1). If $A=S^1\times S^1$, h' fixes $S^1\times S^1$ and $S^1\times S^2-A=R^2\times S^1\cup R^2\times S^1$ [5]. The covering $A\to K$ is at least 2-to-1. If it is actually 2-to-1, then $p^{-1}(K)=A\cup B$, where B is another $S^1\times S^1$. Then B is contained in one of the two $R^2\times S^1$. But h' interchanges two $R^2\times S^1$, while $h'(p^{-1}(K))=p^{-1}(K)$ and therefore $h'(B)=B$ which is impossible. Hence $p^{-1}(K)=A$. Hence each of the two $R^2\times S^1$ double cover $P_3\# P_3-K$. Hence we have an exact sequence

$$0\longrightarrow Z\xrightarrow{f}\pi_1(M-U)\xrightarrow{g}Z_2\longrightarrow 0.$$

Hence if we choose $x\in\pi_1(M-U)$ such that $g(x)\neq 0$ and a generator y in Z, then $\pi_1(M-U)$ is generated by x and y. Suppose $xyx^{-1}=y^{-1}$. By abelianizing we have $y^2=1$. And since x is of finite order too, $H_1(M-U)$ is finite which contradicts the fact that $H_1(M-U)$ is of rank 1. Hence $\pi_1(M-U)$ is abelian, and hence $\pi_1(M-U)=Z+$torsion part. But since we have a universal covering $R^2\times R^1\to M-U$ and no nontrivial finite group can act freely on a finite-dimensional, contractible space, $\pi_1(M-U)=Z$.

Case (2). If $A=K$, then h' fixes only A and $S^1\times S^2-A=R^2\times S^1$. If $p^{-1}(K)$ had any other component B, then $B\approx K$ and $B\subseteq R^2\times S^1\subseteq R^3$. But K cannot be embedded in R^3. Hence $p^{-1}(K)=A$. Then $R^3-A\approx R^2\times S^1$ covers $P_3\# P_3$ 4-to-1. Hence

$$0\longrightarrow Z\longrightarrow\pi_1(M-U)\xrightarrow{\zeta}Z_2+Z_2\longrightarrow 0$$

is exact, or

$$0\longrightarrow Z\longrightarrow\pi_1(M-U)\xrightarrow{\zeta}Z_4\longrightarrow 0$$

is exact. Let N be a subgroup of Z_2+Z_2 (or Z_4) such that $N\approx Z_2$. Let $p':X\to M-U$ be the double covering corresponding to $\zeta^{-1}(N)$. Then there exists a double covering $p'':R^2\times S^1\to X$. Then the argument in Case (1) shows that $\pi_1X\approx Z$. Now consider the double covering $p':X\to M-U$. Repeating the same argument in Case (1), we get $\pi_1(M-U)=Z$.

Since $M-U$ is an irreducible, orientable, compact 3-manifold with $\pi_1(M-U)=Z$ and $\mathrm{Bd}(M-U)$ is homeomorphic to $S^1\times S^1$, $M-U$ is homeomorphic to $D^2\times S^1$. This proves the theorem.

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