ON A CONSTRUCTION OF BREDON

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ABSTRACT. Using a homotopy-theoretical description of a geometric pairing due to Bredon, we show how to rederive Bredon’s results on the pairing. Furthermore, we are able to, in some sense, complete these results by combining the homotopy-theoretical approach with Sullivan’s determination of the 2-primary Postnikov decomposition of the space $G/PL$.

1. Introduction. In [1], Bredon introduced a geometric pairing

$$\Gamma_n \times \Pi_{n+k}(S^n) \xrightarrow{\rho_{n,k}} \Gamma_{n+k},$$

$\Gamma_i$ being the group of differential structures on the $i$-sphere$^2$ and used this pairing to construct certain semifree actions of groups on spheres. There is also the pairing

$$\Pi_n(PL/O) \times \Pi_{n+k}(S^n) \xrightarrow{\gamma_{n,k}} \Pi_{n+k}(PL/O)$$
defined by composition, $PL/O$ being the fibre of the natural map $BO \to BPL$ of classifying spaces. Since, by smoothing theory, $\Pi_{n}(PL/O)$ and $\Gamma_n$ are isomorphic groups, it is natural to inquire about the relation between these two pairings. In fact, as observed by Bredon [1] and the present author [3], these two pairings coincide.

While Bredon’s geometric definition is rather natural and simple, the main results and proofs are more perspicuous using the homotopy-theoretical definition. Thus, in §2 we show how to retrieve (and slightly generalize) the theorems of [1, §§1, 2] in this context, in particular the following important result, which we state as

**Theorem A** [1, Theorem 2.1]. Let $x \in \Pi_{n+k}(S^n)$, $\sigma \in \Gamma_n$ and let $p_i(\sigma) \subset \Pi_{n+i}(S^i)$ denote the set of all elements represented (via the Pontryagin-Thom construction) by framed embeddings $\Sigma^n \times D^i \subset S^{n+i}$, $\Sigma^n$
a homotopy sphere representing \( \sigma \). Then
\[
p_{i}(\sigma) \circ \Sigma^{i} \alpha \subseteq p_{i}(\rho_{n,k}(\sigma, \alpha)),
\]
where \( \Sigma^{i} \alpha \) is the \( i \)th iterated suspension of \( \alpha \).

However, the main contribution of this note is the following Theorem B (essentially conjectured by Bredon [1, Corollary 2.3 and succeeding remarks]) which, taken together with Theorem A, gives a rather good hold on Bredon’s pairing.

**Theorem B.** If \( bP_{n+1} \) is the subset of \( \Gamma_{n} \) consisting of those homotopy spheres which bound \( \Pi \)-manifolds, then the pairing \( \rho_{n,k} \) restricts to a pairing \( \tilde{\rho}_{n,k}:bP_{n+1} \times \Pi_{n+k}(S^{n}) \rightarrow bP_{n+k+1} \). Moreover, \( \tilde{\rho}_{n,k} \) is trivial for \( k>0 \).

The proof of Theorem B, which relies on results of Sullivan [4], will also be carried out in §2.

As a final introductory comment, we remark that while the main interest in the composition pairing \( \Pi_{n}(PL/O) \times \Pi_{n+k}(S^{n}) \rightarrow \Pi_{n+k}(PL/O) \) lies in its geometric interpretation, the pairing has also proved to be of some use in studying the \( k \)-invariants of \( PL/O \) ([5], [3]).

2. The main properties of \( \rho_{n,k} \). In this section we deduce the main properties of the pairing \( \rho_{n,k} \), working in the homotopy-theoretical context. We begin with a proposition which summarizes and slightly generalizes the results of [1, §1].

**Proposition.** The pairing \( \rho_{n,k}:\Gamma_{n} \times \Pi_{n+k}(S^{n}) \rightarrow \Gamma_{n+k} \) is bilinear and associative in the sense that the diagram
\[
\begin{array}{ccc}
\Gamma_{n} \times \Pi_{n+k}(S^{n}) \times \Pi_{n+k+1}(S^{n+k}) & \xrightarrow{id \times \text{comp}} & \Gamma_{n} \times \Pi_{n+k+1}(S^{n}) \\
\downarrow \rho_{n,k} \times id & & \downarrow \rho_{n,k+1} \\
\Gamma_{n+k} \times \Pi_{n+k+1}(S^{n+k}) & \xrightarrow{\rho_{n,k+1}} & \Gamma_{n+k+1}
\end{array}
\]
is commutative.

**Proof.** Using the identification of \( \rho_{n,k} \) with \( \gamma_{n,k} \), all these statements, except perhaps the linearity in the first variable, are trivial. As for linearity in the first variable, this follows from the fact that \( PL/O \) is an \( H \)-space.

We remark that linearity in the first variable was proved in [1] only under the assumption that \( \Pi_{n+k}(S^{n}) \) is stable, i.e. \( k<n-1 \). Our proof, exploiting the \( H \)-structure on \( PL/O \), shows this restriction on \( k \) to be unnecessary.

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3 The bilinearity of \( \gamma_{n,k} \) has been independently observed by Schultz (Smooth structures on \( S^{n} \times S^{n} \), Ann. of Math. (2) 90 (1969), 187-198) in a closely related context.
We come now to the two key results, Theorems A and B.

**Proof of Theorem A.** Let \( \Gamma'_{n+1} \) denote the set of framed embeddings of homotopy \( n \)-spheres in \( S^{n+1} \), \( G_t \) the space of maps \( S^{n+1} \to S^{n+1} \) of degree 1. In addition to the obvious (forgetful) map \( q_i: \Gamma'_{n+1} \to \Gamma_t \), there is a map \( \omega_i: \Gamma'_{n+1} \to \Pi_i(G_t) \) described in [2]. It is not difficult to see that the set \( p_i(\sigma) \), as originally defined by Kervaire-Milnor, can equivalently be described as \( J_n(\omega_n(\varphi^{-1}_n(\sigma))) \), where \( J_n: \Pi_n(G_t) \to \Pi_{n+1}(S^n) \) is obtained, as usual, by the Hopf construction. Thus, if \( \beta \in p_i(\sigma) \), there exists \( \eta \in \omega_{n+1}(\varphi^{-1}_n(\sigma)) \) such that \( \beta = J_n(\eta) \). But by a known formula (cf. [1, p. 442]), we have \( J_n(\eta) \circ \Sigma \sigma = J_{n+1}(\eta \circ \sigma) \). Moreover, it is not difficult to see, for example by using the homotopy-theoretical interpretation of \( \Gamma'_{n+1} \), \( q_i \), \( \omega_i \) (see footnote 4) that \( \eta \circ \sigma \in \omega_{n+1}(\varphi^{-1}_{n+1}(\rho_{n,k}(\sigma, \sigma))) \) and the theorem follows.

**Proof of Theorem B.** If \( G_k \) is the set of maps \( S^{k-1} \to S^{k-1} \) of degree \( \pm 1 \) and \( G = \lim_{k \to \infty} G_k \), there are natural maps \( O \to G \) and a fibration \( PL/O \to G/O \to G/PL \). For \( \alpha \in \Pi_{n+k}(S^n) \), we consider the diagram.

\[
\begin{array}{ccc}
\Pi_{n+1}(G/PL) & \xrightarrow{\partial_n} & \Pi_n(PL/O) \\
\downarrow & & \downarrow \\
\Pi_{n+k+1}(G/PL) & \xrightarrow{\partial_{n+k}} & \Pi_{n+k}(PL/O)
\end{array}
\]

(2.1)

The commutativity of (2.1) is a consequence of a well-known formula in fibre-space theory. From [4], we know that \( bP_{n+1} \) can be described homotopy-theoretically as the image of \( \partial_3: \Pi_{n+1}(G/PL) \to \Pi_4(PL/O) \), so to prove the theorem, it is sufficient to prove the left-hand vertical arrow in (2.1) is the zero map.

To this end, observe first that since \( k > 0 \), \( \Sigma \alpha \) has finite order. We may further clearly assume \( \Sigma \alpha \) has prime-power order \( p^j \) and distinguish two cases, according as \( p \) is odd or \( p = 2 \).

For \( p \) odd, the conclusion is trivial because the only torsion in \( \Pi_{n+k}(G/PL) \) is of order 2 [4]. If \( p = 2 \), we argue as follows. Let \( Y \) be the space \( (K(Z_2, 2) \times_{8S^4} K(Z, 4)) \times \Pi_{\geq 3} K(Z_2, 4i - 2) \times K(Z, 4i) \). Sullivan [4] shows that after localizing at the prime 2, \( Y \) and \( G/PL \) become homotopy equivalent. We are therefore reduced to proving that \( \Pi_{n+k+1}(Y) \xrightarrow{\Sigma \alpha} \Pi_{n+k+1}(Y) \) is the zero map, which is evident from the structure of \( Y \).

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4 See Rourke-Sanderson, *Block bundles*, III, Ann. of Math. (2) 87 (1968), 431–483, for a homotopy-theoretic description of the set \( \Gamma'_{n+1} \) and the maps \( q_i, \omega_i \).

5 It is actually true that the composition pairing \( \Pi_{n+1}(G/PL) \times \Pi_{n+k+1}(S^{n+1}) \to \Pi_{n+k+1}(G/PL) \) is trivial but we do not need this additional fact.
Bibliography


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