ON OSCILLATIONS FOR SOLUTIONS OF
nTH ORDER DIFFERENTIAL EQUATIONS

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Abstract. Necessary and sufficient conditions are given that all
solutions of \( x^{(n)} + f(t, x, x', \ldots, x^{(n-2)}) = 0 \) are oscillatory for \( n \) even and are oscillatory or tend monotonically to zero as \( t \to \infty \) for
\( n \) odd. The results generalize recent results of J. S. W. Wong and
G. H. Ryder and D. V. V. Wend.

1. Introduction. In this paper we are dealing with differential equations
of the form
\[
\tag{\ast} x^{(n)} + f(t, x, x', \ldots, x^{(n-2)}) = 0,
\]
where \( f \) is continuous in \([a, \infty) \times R^{n-1}, a \geq 0\).

We consider only nontrivial solutions of (\ast) which are indefinitely con-
tinuable to the right. A solution of (\ast) is said to be oscillatory if it has
arbitrarily large zeros and nonoscillatory if it is eventually of constant
sign. The equation (\ast) is said to be oscillatory if every solution of (\ast) is
oscillatory. Recently, the present author [1] gave a definition, called
generalized strongly continuous, which is a generalization of Wong’s [4]
definition and discussed the oscillatory properties of (\ast). A function
\( f(t, x_1, \ldots, x_{n-1}) \) is called generalized strongly continuous from the left
at \( x_{1c} \) if \( f(t, x_1, \ldots, x_{n-1}) \) is continuous in \([a, \infty) \times R^{n-1}, a \geq 0\), and for
each \( \varepsilon > 0 \) there exists \( \delta > 0, T \geq 0 \) and \( x_c \in [x_{1c} - \delta, x_{1c}] \) such that for all
\( x_i \in [x_{1c} - \delta, x_{1c}] \), for all \( x_i \) satisfying \( |x_i - k_i| \leq \delta \) (\( k_i \) are arbitrary real
constants) for \( i = 2, \ldots, n-1 \), and for all \( t \geq T \),
\[
(1 - \varepsilon)f(t, x_0, k_2, \ldots, k_{n-1}) \leq f(t, x_1, \ldots, x_{n-1}) \leq (1 + \varepsilon)f(t, x_{1c}, k_2, \ldots, k_{n-1}).
\]

Generalized strong continuity from the right at \( x_{1c} \) is defined analogously.
A function \( f(t, x_1, \ldots, x_{n-1}) \) is said to be generalized strongly continuous
at \( x_{1c} \) if it is generalized strongly continuous both from the left and from
the right at \( x_{1c} \).

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The purpose of this paper is to extend Wong’s [4, Theorem 4] result to the arbitrary nth order equation (*). The results of this paper also extend recent results of Ryder and Wend [3].

For convenience in stating our theorems, we list the following conditions:

(1) $x_1 f(t, x_1, \ldots, x_{n-1}) \geq 0 \quad (x_1 \neq 0),$

$y^{(n)} + f(t, y, y', \ldots, y^{(n-2)}, y^{(n-1)}) = 0,$

where $f$ is continuous in $[a, \infty) \times \mathbb{R}^n$, $a \geq 0$,

(3) there exists constants $c (\neq 0)$ and $k_2, \ldots, k_{n-1}$ such that

$$\left| \int_{t}^{\infty} t^{n-1} f(t, c, k_2, \ldots, k_{n-1}) \, dt \right| < \infty.$$  

2. Nonoscillation theorems.

Theorem A [1]. Assume that $n$ is even and that condition (1) holds. Let $f(t, x_1, \ldots, x_{n-1})$ be generalized strongly continuous from the left for $x_1 > 0$ and generalized strongly continuous from the right for $x_1 < 0$. Then condition (3) is a necessary and sufficient condition for equation (*) to have a bounded nonoscillatory solution.

Theorem 1. Assume that $n$ is odd and that condition (1) holds. Let $f(t, x_1, \ldots, x_{n-1})$ be generalized strongly continuous.

Then condition (3) is a necessary and sufficient condition for equation (*) to have a bounded nonoscillatory solution which does not tend monotonically to zero as $t \to \infty$.

Proof of Theorem 1. Suppose $y(t)$ is a bounded nonoscillatory solution which does not tend monotonically to zero as $t \to \infty$. Assume without loss of generality, $y(t) > 0$. As in the proof of Theorem 1 of Ryder and Wend [3], by the boundedness of $y(t)$, we have the following:

(4) $\lim_{t \to \infty} y^{(i)}(t) = 0 \quad (i = 1, 2, \ldots, n - 1),$

(5) $-y'(t) = \int_{t}^{\infty} \frac{(u - t)^{n-2}}{(n - 2)!} f(u, y(u), y'(u), \ldots, y^{(n-2)}(u)) \, du \geq 0.$

By (5), $y(t)$ decreases to a limit $L \geq 0$ and by the assumptions on $y(t)$, $L > 0$. Integrating (5) from $T$ to $\infty$, we have

(6) $\int_{T}^{\infty} \frac{(u - T)^{n-1}}{(n - 1)!} f(u, y(u), y'(u), \ldots, y^{(n-2)}(u)) \, du.$
The generalized strong continuity of \( f(t, x_1, x_2, \ldots, x_{n-1}) \) implies that for \( \varepsilon = \frac{1}{2} \) there exist \( \delta > 0 \), \( T \geq 0 \) and \( L_\delta \in [L, L + \delta] \) such that for all \( x_i \) satisfying \( |x_i - k_i| \leq \delta \) \((i = 2, \ldots, n-1)\), and for all \( x_1 \in [L, L + \delta] \) and \( t \geq T \),

\[
f(t, x_1, x_2, \ldots, x_{n-1}) \geq \frac{1}{2} f(t, L_\delta, k_2, \ldots, k_{n-1}).
\]

From (4) it follows that there exists a \( T_0 \geq T \) sufficiently large that for all \( t \geq T_0 \), \( x(t) \) satisfies \( L \leq x(t) \leq L + \delta \) and

\[
|x^{(i)}(t) - 0| \leq \delta \quad (i = 1, 2, \ldots, n - 2).
\]

Thus we obtain, for \( t \geq T_0 \),

\[
0 < \frac{1}{2} f(t, L_\delta, 0, \ldots, 0) < f(t, x(t), x'(t), \ldots, x^{(n-2)}(t)).
\]

Accordingly, by (6), we have

\[
x(t) \geq \frac{1}{2} \int_T^\infty \frac{(u - T)^{n-1}}{(n - 1)!} f(u, L_\delta, 0, \ldots, 0) \, du,
\]

which implies (3).

Conversely, suppose that (3) holds for some constants \( c \neq 0 \) and \( k_i \) \((i = 2, \ldots, n - 1)\). Then we can prove there exists a solution \((x_0(t), \ldots, x_{n-2}(t))\) to the following system of integral equations:

\[
x_{n-2}(t) = k_{n-1} - \int_t^\infty (s - t)f(s, x_0(s), \ldots, x_{n-2}(s)) \, ds,
\]

\[
x_{n-3}(t) = k_{n-2} + \int_t^\infty \frac{(s - t)^2}{2!} f(s, x_0(s), \ldots, x_{n-2}(s)) \, ds,
\]

\[
\vdots
\]

\[
x_0(t) = c + \int_t^\infty \frac{(s - t)^{n-1}}{(n - 1)!} f(s, x_0(s), \ldots, x_{n-2}(s)) \, ds.
\]

(7)

Note that we can obtain

\[
0 < c \leq x_{0,N}(t) \leq c + c/M \quad (M \text{ is sufficiently large}),
\]

by the same argument as in the even case (cf. [2], [4]).

Then, by using the method of successive approximations (cf. [2], [4]),
we get a solution \((x_0(t), \ldots, x_{n-2}(t))\) and it is clear that \( x_0(t) \) is a desired nonoscillatory solution of (*) which does not tend monotonically to zero as \( t \to \infty \).
3. Oscillation theorems.

**Theorem 2.** Assume that $n$ is even and that condition (1) holds. Let $f(t, x_1, x_2, \ldots, x_{n-1})$ be generalized strongly continuous from the left for $x_1 > 0$ and generalized strongly continuous from the right for $x_1 < 0$ and suppose there exists a function $\phi(x)$ with the following properties:

(a) $\phi(x)$ is a nondecreasing continuous function of $x$ satisfying $x\phi(x) > 0$ whenever $x \neq 0$;

(b) there exists $c \neq 0$ and $k_i$ ($i = 2, \ldots, n-1$), such that

$$\lim_{|x| \to \infty} \inf \frac{f(t, x_1, x_2, \ldots, x_{n-1})}{\phi(x_1)} \geq k |f(t, c, k_2, \ldots, k_{n-1})|$$

for some positive constant $k$ and for all $t \geq T$ and

$$\lim_{|x| \to \infty} \left| \int_{x}^{\infty} (du|\phi(u)) \right| < \infty.$$

Then a necessary and sufficient condition for (*) to be oscillatory is that

$$\int_{t}^{\infty} t^{n-1} f(t, c, k_2, \ldots, k_{n-1}) dt = \infty$$

for all constants $c$ ($\neq 0$) and $k_i$ ($i = 2, \ldots, n-1$).

**Proof.** Assume that (8) does not hold. Then (3) holds for some $c \neq 0$ and $k_i$ ($i = 2, \ldots, n-1$). Hence by Theorem A, equation (*) has a bounded nonoscillatory solution, so that condition (8) is necessary.

Conversely, let $x(t) > 0$ be a nonoscillatory solution of (*). In view of Ryder and Wend's [3] arguments, $x(t)$ must be nondecreasing and hence must tend to a limit, finite or infinite. Suppose first that the limit is finite, i.e. $\lim_{t \to \infty} x(t) = L$ ($> 0$). Then, we obtain (cf. [1, Theorem 1]) for some $L_\delta$ and $t$ sufficiently large

$$\int_{t}^{\infty} u^{n-1} f(u, L_\delta, 0, \ldots, 0) du < \infty$$

which contradicts (8).

Next, we turn to the case $\lim_{t \to \infty} x(t) = \infty$. By again using the argument of Ryder and Wend [3], we have for sufficiently large $t$, say $t \geq T$,

$$x'(t) \geq \int_{t}^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} f(u, x(u), x'(u), \ldots, x^{(n-2)}(u)) du$$

for all $t \geq T$.
if \( x'(t), \ldots, x^{(n-1)}(t) \) tend monotonically to zero as \( t \to \infty \), or

\( x'(t) > \int_t^\infty \frac{(t - t_1)^{n-2}}{(n-2)!} f(u, x(u), x'(u), \ldots, x^{(n-2)}(u)) \, du, \tag{10} \)

where \( t_1 < t \) is sufficiently large, if \( x'(t) \) does not tend monotonically to zero as \( t \to \infty \).

In case (9) holds, multiply each side of the inequality in (9) by \((\phi(x))^{-1}\). Since \( \phi(x) \) is nondecreasing in \( x \) and \( x(t) \) is nondecreasing in \( t \), we obtain

\[
\int_t^\infty \frac{f(u, x(u), x'(u), \ldots, x^{(n-2)}(u)) (u - t)^{n-2}}{(n-2)!} \, du \geq 0.
\]

Integrating (11) from \( T \) to \( t \geq T \),

\[
\int_{x(T)}^{x(t)} \frac{du}{\phi(u)} \geq \int_T^t \int_s^\infty \frac{f(u, x(u), x'(u), \ldots, x^{(n-2)}(u)) (n - s)^{n-2}}{(n-2)!} \, d\phi(x) \, ds
\]

By (b1) and the fact that \( \lim_{t \to \infty} x(t) = \infty \), we may also choose \( T \) so large that, for \( t \geq T \),

\[
\frac{f(t, x(t), x'(t), \ldots, x^{(n-2)}(t))}{\phi(x(t))} \geq \frac{k}{2} \left| f(t, c, k_2, \ldots, k_{n-1}) \right| > 0.
\]

Substituting the above inequality in (12) and letting \( t \to \infty \), we obtain a contradiction to (b2) if (8) holds.

In case (10) holds, an argument similar to that above again leads to a contradiction to (b2). This proves the sufficiency part of the theorem.

**Theorem 3.** In addition to the hypotheses (a) and (b) of Theorem 2, assume \( n \) is odd and that condition (1) holds. Let \( f(t, x_1, \ldots, x_{n-1}) \) be generalized strongly continuous. Then condition (8) is a necessary and sufficient condition for the solution of (*) to be oscillatory or tend monotonically to zero as \( t \to \infty \).

**Proof.** The necessity follows from Theorem 1.

To prove the sufficiency, suppose \( x(t) > 0 \) is a nonoscillatory solution of (*) not tending monotonically to zero as \( t \to \infty \).

In this case, we find that \( x(t) \) satisfies either (5) or (10) by the argument used in Ryder and Wend [3]. If \( x(t) \) satisfies (5), then our Theorem 3

...
follows from Theorem 1, and if \( x(t) \) satisfies (10), then it follows as in the proof of Theorem 2.

**Remarks.** For equation (2), Theorem 2 and Theorem 3 remain valid under the more severe condition that \( f \) be bounded for all \( t \) for each \( n \)-tuple \( (x_1, x_2, \cdots, x_n) \) (cf. [2]). But we note here that the proofs of the sufficiency parts of Theorems A, 1, 2 and 3 remain valid for the more general equation (2). Only in the necessity parts does the boundedness of \( f \) in \( t, -\infty < t < \infty \), enter in—in solving the system of integral equations as in [2]. Our results also extend Theorem 2 of Ryder and Wend [3].

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**References**


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