EXTENSIONS OF LEFT UNIFORMLY CONTINUOUS
FUNCTIONS ON A TOPOLOGICAL SEMIGROUP

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ABSTRACT. For any topological semigroup $S$ with separately continuous operation, let $C(S)$ denote the set of all bounded continuous real valued functions on $S$ with the supremum norm and let $LUC(S)$ denote the set of all $f$ in $C(S)$ such that whenever \((s(t))\) is a net in $S$ which converges to some $s$ in $S$, then \(\sup\{|f(s(t)) - f(st)| : t \in S\}\) converges to 0. In this paper we prove that if $S$ is an abelian subsemigroup of a compact topological group and $f \in LUC(S)$, then there is an $F \in LUC(G)$ where $F(s) = f(s)$ for all $s \in S$. We also show whenever there is an extension of the type indicated above, there is a norm preserving extension.

1. Introduction. By a topological semigroup we mean a semigroup with a Hausdorff topology in which the product $st$ is separately continuous. If $X$ is a topological space, $C(X)$ indicates the algebra of all bounded continuous real valued functions on $X$ with the supremum norm. If $S$ is a topological semigroup, $l_s : C(S) \rightarrow C(S)$ is defined by $l_s f(t) = f(st)$ and $r_s : C(S) \rightarrow C(S)$ is defined by $r_s f(t) = f(ts)$. Let $S$ be a topological semigroup. $f \in C(S)$ is said to be left uniformly continuous or $f \in LUC(S)$ when the function $\theta : S \rightarrow C(S)$ defined by $\theta(s) = l_s f$ is continuous on $S$. The space $LUC(S)$ is a closed subalgebra of $C(S)$ which contains the constants [8]. Mitchell [7] considered topological semigroups $T$ and subsemigroups $S$. He showed if $S$ is dense in $T$ and $LUC(S)$ has a left invariant mean, so has $LUC(T)$. In the next section we will show that if $S$ is an abelian subsemigroup of a compact topological group, then every $f \in LUC(S)$ has an extension to an $F \in LUC(T)$.

If $(X, \mathcal{U})$ is a uniform space, then $UC(X)$ denotes the set of all bounded real valued functions on $X$ which are uniformly continuous with respect to $\mathcal{U}$. In the following proposition the equivalence of (a) and (b) follows from
Kelley [5, p. 86], (b) implies (c) is found in Granirer and Lau [2, Lemma 3], the rest comes directly from the definitions.

**Proposition 1.** Let $S$ be a topological semigroup and $f \in C(S)$. The following are equivalent.

(a) $f \in \text{LUC}(S)$.

(b) For every $s \in S$, if $\{s(\gamma)\}$ is a net converging to $s$, then $l_{s(\gamma)} f$ converges to $l_s f$ uniformly.

(c) $\{r_{t}: t \in S\}$ is an equicontinuous family of functions on $S$.

If $S$ is a topological group, then (a), (b) and (c) are equivalent to (d).

(d) $f \in \text{UC}(S)$ where the uniformity on $S$ is the usual right uniformity on a topological group [1, p. 243].

2. Extension theorems. It will be convenient to have the following definition. Let $S$, $T$ be topological semigroups, $S$ a subsemigroup of $T$. Then we will say the pair $(S, T)$ has property $P$, if for every $f \in \text{LUC}(S)$ there is an $F \in \text{LUC}(T)$ such that $F|_S = f$. $F|_S$ indicates the restriction of $F$ to $S$.

**Proposition 2.** Let $G$ be a topological group with the right uniformity and $S$ a subsemigroup of $G$ with the restriction uniformity. The following are equivalent.

(a) The pair $(S, G)$ has property $P$.

(b) $\text{LUC}(S) = \text{UC}(S)$.

(c) $\text{LUC}(S) = \text{UC}(S)$.

**Proof.**

(a)$\rightarrow$(b) If $f \in \text{LUC}(S)$, then $f = F|_S$ where $F \in \text{LUC}(G) = \text{UC}(G)$. Hence $f|_S \in \text{UC}(S)$.

(b)$\rightarrow$(c) If $f \in \text{UC}(S)$, by Katetov [4, Theorem 3], $f|_S = \text{LUC}(G)|_S$. Hence, by Lau [6, Proposition 1.3.1], $f \in \text{LUC}(S)$.

(c)$\rightarrow$(a) If $f \in \text{LUC}(S) = \text{UC}(S)$, by Katetov [4, Theorem 3] there is an $F \in \text{UC}(G) = \text{LUC}(G)$ such that $F|_S = f$.

The following theorem is an immediate consequence of Propositions 1 and 2.

**Theorem 3.** If $G$ is a topological group and $G'$ is a subgroup of $G$, then the pair $(G', G)$ has property $P$.

The following example shows that we do not always have the extension property from a subsemigroup of a topological group to the group.

**Example.** Let $G$ be the real numbers under addition, $S$ be the positive real numbers and $f(x) = \sin(1/x)$. $f$ is not uniformly continuous on $S$ since it does not have a continuous extension to $\text{cl}(S)$. But $f \in \text{LUC}(S)$. So the pair $(S, G)$ does not have property $P$. We can see $f \in \text{LUC}(S)$ since if
\( \{s(n)\} \) is a net in \( S \) converging to \( s \in S \), then

\[
|l_{s(n)}f(x) - l_s f(x)| = \left| \sin \frac{1}{s(n) + x} - \frac{1}{s + x} \right|
\]

\[
= \left| 2 \sin \frac{1}{2} \left( \frac{1}{s(n) + x} - \frac{1}{s + x} \right) \cos \frac{1}{2} \left( \frac{1}{s(n) + x} + \frac{1}{s + x} \right) \right|
\]

\[
\leq 2 \left| \sin \frac{1}{2} \frac{s - s(n)}{(s(n) + x)(s + x)} \right|
\]

But \( s \) is positive, so for sufficiently large \( n \), \( s(n) > \delta \) and \( s > \delta \) where \( \delta > 0 \). Since \( x > 0 \), \( s(n) + x > \delta \) and \( s + x > \delta \). So

\[
\left| \frac{s - s(n)}{(s(n) + x)(s + x)} \right| < \left| \frac{s - s(n)}{\delta^2} \right| \quad \text{for all } x \in S.
\]

Since the sine function is continuous and \( (s - s(n)) / \delta^2 \) converges to 0, there is an \( n_0 \) such that for all \( n \geq n_0 \),

\[
2 \left| \sin \frac{1}{2} \frac{s - s(n)}{(s(n) + x)(s + x)} \right| < \epsilon \quad \text{for all } x \in S.
\]

So \( f \in \text{LUC}(S) \).

Mitchell [7, pp. 640–641] has given an example of a compact topological semigroup \( T \) and a dense subsemigroup \( S \) of \( T \) where the pair \( (S, T) \) does not have property \( P \). We will now show that we have the desired extension whenever \( T \) is a compact topological group and \( S \) is abelian.

**Theorem 4.** If \( G \) is a compact topological group and \( S \) is an abelian subsemigroup of \( G \), then the pair \( (S, G) \) has property \( P \).

**Proof.** (i) Since \( \text{cl}(S) \) is a compact cancellation semigroup, it is a topological group [3, Theorem 9.16]. Hence by Theorem 3 we may assume \( \text{cl}(S) = G \).

(ii) Let \( f \in \text{LUC}(S) \), \( x(a) \) be a net in \( S \) which converges to \( e \) and \( a \in S \). Then \( x(a)a \) converges to \( a \). Hence \( \lim_{a} f(x(a)y) = f(ay) \) uniformly for all \( y \) in \( S \).

(iii) Now since \( G \) is compact, there is a unique uniformity for \( G \). This consists of all neighborhoods of the diagonal \( \Delta \) [5, p. 197]. Let \( A_{\epsilon} = \{(x, y) \in S \times S : |f(x) - f(y)| \geq \epsilon \} \). Then \( f \) is uniformly continuous on \( S \) iff for every \( \epsilon > 0 \), \( \text{cl}(A_{\epsilon}) \cap \Delta = \emptyset \). Suppose there is an \( f \in \text{LUC}(S) \) where \( f \notin \text{UC}(S) \). Then there is an \( \epsilon > 0 \) such that \( \text{cl}(A_{\epsilon}) \cap \Delta \neq \emptyset \). Hence there is a net \( (x(\alpha), y(\alpha)) \) in \( A_{\epsilon} \) which converges to \( (t, t) \in \Delta \). Now let \( \{t(\beta)\} \) be a net in \( S \) which converges to \( t^{-1} \). Then \( \lim_{\alpha, \beta} (x(\alpha)t(\beta)) = e = \lim_{\alpha} (y(\alpha)t(\beta)) \). Then by part (ii) above given \( \epsilon > 0 \) there are \( \alpha \) and \( \beta \) such that

\[
|f(x(\alpha)t(\beta)s) - f(s)| < \epsilon/2 \quad \text{and} \quad |f(y(\alpha)t(\beta)s) - f(s)| < \epsilon/2
\]
for all $s \in S$. But $x(x) \in S$ and $y(z) \in S$ hence
\[
|f(x(x)) - f'(y(z))| \leq |f(x(z)) - f'(y(z))| + |f(x(z)) - f'(y(z))| < \varepsilon.
\]
This contradicts that $(x(x), y(z)) \in A_e$.

**Corollary 5.** If $G$ is a topological group, $S$ is an abelian subsemigroup of $G$ where $\text{cl}(S)$ is compact, then the pair $(S, G)$ has property $P$.

**Proof.** $\text{cl}(S)$ is a compact topological group [3, Theorem 9.16]. So we can make the extension from $S$ to $\text{cl}(S)$. Then by Theorem 3 we can make the extension from $\text{cl}(S)$ to $G$.

Finally we will show that whenever we have an extension, we have a norm preserving extension.

**Proposition 6.** If $S$ is a subsemigroup of a topological semigroup $T$ and the pair $(S, T)$ has property $P$, then given $f \in \text{LUC}(S)$ there is a $g \in \text{LUC}(T)$ such that $g|_S = f$ and $\|g\| = \|f\|$.

**Proof.** Let $F \in \text{LUC}(T)$ be an extension of $f \in \text{LUC}(S)$. If $g = (F\|f\|)\nu (-\|f\|)$ where $\|f\|$ and $-\|f\|$ are constant functions, then $g \in \text{LUC}(T)$ [8, Lemma 1.1] and is the desired norm preserving extension of $f$.

**Bibliography**


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