EXTENSIONS OF LEFT UNIFORMLY CONTINUOUS FUNCTIONS ON A TOPOLOGICAL SEMIGROUP

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Abstract. For any topological semigroup S with separately continuous operation, let C(S) denote the set of all bounded continuous real valued functions on S with the supremum norm and let LUC(S) denote the set of all f in C(S) such that whenever \((s(t))\) is a net in S which converges to some s in S, then \(\sup\{|f(s(t))−f(st)| : t \in S\}\) converges to 0. In this paper we prove that if S is an abelian subsemigroup of a compact topological group and \(f \in \text{LUC}(S)\), then there is an \(F \in \text{LUC}(G)\) where \(F(s) = f(s)\) for all \(s \in S\). We also show whenever there is an extension of the type indicated above, there is a norm preserving extension.

1. Introduction. By a topological semigroup we mean a semigroup with a Hausdorff topology in which the product st is separately continuous. If \(X\) is a topological space, \(C(X)\) indicates the algebra of all bounded continuous real valued functions on \(X\) with the supremum norm. If \(S\) is a topological semigroup, \(l_s: C(S) \rightarrow C(S)\) is defined by \(l_sf(t) = f(st)\) and \(r_s: C(S) \rightarrow C(S)\) is defined by \(r_sf(t) = f(ts)\). Let \(S\) be a topological semigroup. \(f \in C(S)\) is said to be left uniformly continuous or \(f \in \text{LUC}(S)\) when the function \(l_s: S \rightarrow C(S)\) defined by \(l_s = l_sf\) is continuous on \(S\). The space \(\text{LUC}(S)\) is a closed subalgebra of \(C(S)\) which contains the constants [8]. Mitchell [7] considered topological semigroups \(T\) and subsemigroups \(S\). He showed if \(S\) is dense in \(T\) and \(\text{LUC}(S)\) has a left invariant mean, so has \(\text{LUC}(T)\).

In the next section we will show that if \(S\) is an abelian subsemigroup of a compact topological group, then every \(f \in \text{LUC}(S)\) has an extension to an \(F \in \text{LUC}(T)\).

If \((X, \mathcal{U})\) is a uniform space, then \(\text{UC}(X)\) denotes the set of all bounded real valued functions on \(X\) which are uniformly continuous with respect to \(\mathcal{U}\). In the following proposition the equivalence of (a) and (b) follows from...
Kelley [5, p. 86], (b) implies (c) is found in Granirer and Lau [2, Lemma 3], the rest comes directly from the definitions.

**Proposition 1. Let** \( S \) **be a topological semigroup and** \( f \in C(S) \). **The following are equivalent.**

(a) \( f \in LUC(S) \).

(b) For every \( s \in S \), if \( \{s(\gamma)\} \) is a net converging to \( s \), then \( l_s f \) converges to \( l_s f \) uniformly.

(c) \( \{r_s f : \gamma \in S\} \) is an equicontinuous family of functions on \( S \).

   If \( S \) is a topological group, then (a), (b) and (c) are equivalent to (d).

(d) \( f \in UC(S) \) where the uniformity on \( S \) is the usual right uniformity on a topological group [1, p. 243].

2. **Extension theorems.** It will be convenient to have the following definition. Let \( S, T \) be topological semigroups, \( S \) a subsemigroup of \( T \). Then we will say the pair \( (S, T) \) has property \( P \), if for every \( f \in LUC(S) \) there is an \( F \in LUC(T) \) such that \( F|_S = f \). \( F|_S \) indicates the restriction of \( F \) to \( S \).

**Proposition 2. Let** \( G \) **be a topological group with the right uniformity and** \( S \) **a subsemigroup of** \( G \) **with the restriction uniformity. The following are equivalent.**

(a) The pair \( (S, G) \) has property \( P \).

(b) \( LUC(S) = UC(S) \).

(c) \( LUC(S) = UC(S) \).

**Proof.** (a) \( \Rightarrow \) (b) If \( f \in LUC(S) \), then \( f = F|_S \) where \( F \in LUC(G) = UC(G) \). Hence \( f \in UC(G)|_S \subset UC(S) \).

(b) \( \Rightarrow \) (c) If \( f \in UC(S) \), by Katetov [4, Theorem 3], \( f \in UC(G)|_S = LUC(G)|_S \). Hence, by Lau [6, Proposition 1.3.1], \( f \in LUC(S) \).

(c) \( \Rightarrow \) (a) If \( f \in LUC(S) = UC(S) \), by Katetov [4, Theorem 3] there is an \( F \in UC(G) = LUC(G) \) such that \( F|_S = f \).

The following theorem is an immediate consequence of Propositions 1 and 2.

**Theorem 3. If** \( G \) **is a topological group and** \( G' \) **is a subgroup of** \( G \), **then the pair** \( (G', G) \) **has property** \( P \).

The following example shows that we do not always have the extension property from a subsemigroup of a topological group to the group.

**Example.** Let \( G \) be the real numbers under addition, \( S \) be the positive real numbers and \( f(x) = \sin(1/x) \). \( f \) is not uniformly continuous on \( S \) since it does not have a continuous extension to \( cl(S) \). But \( f \in LUC(S) \). So the pair \( (S, G) \) does not have property \( P \). We can see \( f \in LUC(S) \) since if
\{s(n)\} is a net in $S$ converging to $s \in S$, then

$$\lim_{n \to \infty} f(x) - f(x) = \left| \sin \frac{1}{s(n) + x} - \sin \frac{1}{s + x} \right|$$

$$= \left| 2 \sin \frac{1}{2 \left( s(n) + x \right)} \frac{1}{s + x} \right| \cos \left( \frac{1}{2 \left( s(n) + x \right)} + \frac{1}{2 \left( s + x \right)} \right)$$

$$\leq \frac{2 \sin \frac{1}{2 \left( s(n) + x \right)}}{s - s(n)} \frac{s - s(n)}{(s(n) + x)(s + x)}$$

But $s$ is positive, so for sufficiently large $n$, $s(n) > \delta$ and $s > \delta$ where $\delta > 0$. Since $x > 0$, $s(n) + x > \delta$ and $s + x > \delta$. So

$$\frac{s - s(n)}{(s(n) + x)(s + x)} < \left| \frac{s - s(n)}{\delta^2} \right| \quad \text{for all } x \in S.$$

Since the sine function is continuous and $(s - s(n))/\delta^2$ converges to 0, there is an $n_0$ such that for all $n \geq n_0$,

$$2 \sin \frac{1}{2 \left( s(n) + x \right)} \frac{s - s(n)}{(s(n) + x)(s + x)} < \varepsilon \quad \text{for all } x \in S.$$

So $f \in \text{LUC}(S)$.

Mitchell [7, pp. 640–641] has given an example of a compact topological semigroup $T$ and a dense subsemigroup $S$ of $T$ where the pair $(S, T)$ does not have property $P$. We will now show that we have the desired extension whenever $T$ is a compact topological group and $S$ is abelian.

**Theorem 4.** If $G$ is a compact topological group and $S$ is an abelian subsemigroup of $G$, then the pair $(S, G)$ has property $P$.

**Proof.** (i) Since $\text{cl}(S)$ is a compact cancellation semigroup, it is a topological group [3, Theorem 9.16]. Hence by Theorem 3 we may assume $\text{cl}(S) = G$.

(ii) Let $f \in \text{LUC}(S)$, $x(\alpha)$ be a net in $S$ which converges to $e$ and $a \in S$. Then $x(\alpha)a$ converges to $a$. Hence $\lim_{\alpha} f(x(\alpha)y) = f(ay)$ uniformly for all $y$ in $S$.

(iii) Now since $G$ is compact, there is a unique uniformity for $G$. This consists of all neighborhoods of the diagonal $\Delta$ [5, p. 197]. Let $A_x = \{(x, y) \in S \times S : |f(x) - f(y)| \geq \varepsilon \}$. Then $f$ is uniformly continuous on $S$ iff for every $\varepsilon > 0$, $\text{cl}(A_x) \cap \Delta = \emptyset$. Suppose there is an $f \in \text{LUC}(S)$ where $f \notin \text{LUC}(S)$. Then there is an $\varepsilon > 0$ such that $\text{cl}(A_x) \cap \Delta \neq \emptyset$. Hence there is a net $(x(\alpha), y(\beta))$ in $A_x$ which converges to $(t, t) \in \Delta$. Now let $\{t(\beta)\}$ be a net in $S$ which converges to $t^{-1}$. Then $\lim_{\beta} (x(\alpha)t(\beta)) = e = \lim_{\beta} (y(\alpha)t(\beta))$. Then by part (ii) above given $\varepsilon > 0$ there are $\alpha$ and $\beta$ such that

$$|f(x(\alpha)t(\beta)) - f(s)| < \varepsilon/2 \quad \text{and} \quad |f(y(\alpha)t(\beta)) - f(s)| < \varepsilon/2.$$
for all $s \in S$. But $x(s) \in S$ and $y(s) \in S$ hence

$$|f(x(s)) - f(y(s))| \leq |f(x(s)) - f(y(s \cdot (y(s)))x(s))|$$

$$+ |f(x(s) \cdot (y(s)))y(s)) - f(y(s))| < \varepsilon.$$  

This contradicts that $(x(s), y(s)) \in \mathcal{A}_e$.

**Corollary 5.** If $G$ is a topological group, $S$ is an abelian subsemigroup of $G$ where $\text{cl}(S)$ is compact, then the pair $(S, G)$ has property $P$.

**Proof.** $\text{cl}(S)$ is a compact topological group [3, Theorem 9.16]. So we can make the extension from $S$ to $\text{cl}(S)$. Then by Theorem 3 we can make the extension from $\text{cl}(S)$ to $G$.

Finally we will show that whenever we have an extension, we have a norm preserving extension.

**Proposition 6.** If $S$ is a subsemigroup of a topological semigroup $T$ and the pair $(S, T)$ has property $P$, then given $f \in \text{LUC}(S)$ there is a $g \in \text{LUC}(T)$ such that $g|_S = f$ and $\|g\| = \|f\|.$

**Proof.** Let $F \in \text{LUC}(T)$ be an extension of $f \in \text{LUC}(S)$. If $g = (F \|f\|) \vee (-\|f\|)$ where $\|f\|$ and $-\|f\|$ are constant functions, then $g \in \text{LUC}(T)$ [8, Lemma 1.1] and is the desired norm preserving extension of $f$.

**Bibliography**


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