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LINEAR ISOTROPY GROUP OF AN AFFINE SYMMETRIC SPACE

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ABSTRACT. Let K be a subgroup of the general linear group $\mathrm{GL}(n)$. The author found a necessary and sufficient condition that there exist an n-dimensional simply connected affine symmetric space M such that K coincides with the linear isotropy group of all affine automorphisms of M at some point in M.

Let M be an n-dimensional manifold with affine connection, $A(M)$ the group of all affine automorphisms of M, H_p the subgroup of $A(M)$ consisting of all elements of $A(M)$ which fix a point p in M, and dH_p the linear isotropy group determined by H_p. Let V be an n-dimensional vector space, $\mathrm{GL}(n)$ the general linear group of V, and K a subgroup of $\mathrm{GL}(n)$. We shall find a necessary and sufficient condition that there exists a simply connected affine symmetric space M such that K coincides with the linear isotropy group dH_p at some point p in M. We discussed similar problems for a Riemannian symmetric space [6]. First of all we shall prove the following:

LEMMA. *Let T be a tensor in $V\otimes V^*\otimes V^*\otimes V^*$ which satisfies the following conditions.*

(1) $T^i_{\cdot jkl} = -T^i_{\cdot jlk}$,

(2) $T^i_{\cdot jkl} + T^i_{\cdot klj} + T^i_{\cdot ljk} = 0$,

(3) $T^i_{\cdot hmn}T^h_{\cdot jkl} - T^h_{\cdot jmn}T^i_{\cdot hkl} - T^h_{\cdot kmn}T^i_{\cdot jhl} - T^h_{\cdot lmn}T^i_{\cdot jkh} = 0$,

where $T^i_{\cdot jkl}$ are the components of T. Then there is an affine symmetric space whose curvature tensor at some point of it coincides with T.

PROOF. We integrate the following differential equations.

$$\partial\bar{\omega}^i/\partial t = da^i + a^k\bar{\omega}^i_k, \qquad \partial\bar{\omega}^i_k/\partial t = T^i_{\cdot kjl}a^j\bar{\omega}^l,$$

with initial conditions $(\bar{\omega}^i)_{t=0}=0$, $(\bar{\omega}^i_k)_{t=0}=0$.

The solutions $\bar{\omega}^i$, $\bar{\omega}^i_k$ are linear forms in $da^1, \cdots, da^n$ whose coefficients are integral functions of $t, a^1, \cdots, a^n$. If we set $t=1$ and replace a^i by x^i, we have forms $\omega^i(x, dx)$, $\omega^i_j(x, dx)$. Since the determinant of the

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coefficients of $dx^1, \cdots, dx^n$ in $\omega^1, \cdots, \omega^n$ is equal to 1 for $x^1=0, \cdots, x^n=0$, we find a positive number ε such that this determinant is different from zero for $|x^i|<\varepsilon$ $(i=1, 2, \cdots, n)$. Therefore we find an analytic manifold M with analytic forms ω^i, ω^i_j of which $\omega^1, \cdots, \omega^n$ are linearly independent. We shall show that the forms ω^i, ω^i_j satisfy the structure equations

$$d\omega^i = \omega^k \wedge \omega^i_k, \qquad d\omega^i_j = \omega^l_j \wedge \omega^i_l + \tfrac{1}{2}T^i_{\cdot jkl}\omega^k \wedge \omega^l.$$

It is sufficient to prove that these equations are satisfied by the forms $\bar{\omega}^i(t, a; da)$ and $\bar{\omega}^i_j(t, a; da)$ where $d\bar{\omega}^i$, $d\bar{\omega}^i_j$ are calculated regarding t as a constant. If we pose

$$d\bar{\omega}^i = \bar{\omega}^k \wedge \bar{\omega}^i_k + \varepsilon^i, \qquad d\bar{\omega}^i_j = \bar{\omega}^l_j \wedge \bar{\omega}^i_l + \tfrac{1}{2}T^i_{\cdot jkl}\bar{\omega}^k \wedge \bar{\omega}^l + \varepsilon^i_j,$$

ε^i, ε^i_j are quadratic differential forms in $da^1, \cdots, da^n$ and vanish for $t=0$. We shall prove the following equations

$$\partial\varepsilon^i/\partial t = a^k\varepsilon^i_k, \qquad \partial\varepsilon^i_k/\partial t = T^i_{\cdot kjl}a^j\varepsilon^l,$$

which show that $\varepsilon^i=0$, $\varepsilon^i_k=0$ for all $t\in \boldsymbol{R}$.

From $d(\partial\bar{\omega}^i/\partial t)=(\partial/\partial t)\, d\bar{\omega}^i$, we find

$$\partial\varepsilon^i/\partial t = a^k\varepsilon^i_k + \tfrac{1}{2}a^k\bar{\omega}^j \wedge \bar{\omega}^h(T^i_{\cdot kjh} - 2T^i_{\cdot jkh}).$$

But by (1) and (2) we get $T^i_{\cdot kjh}-2T^i_{\cdot jkh}=-2T^i_{\cdot(j|k|h)}$. Therefore we have $\partial\varepsilon^i/\partial t=a^k\varepsilon^i_k$. From $d(\partial\bar{\omega}^i_k/\partial t)=(\partial/\partial t)\, d\bar{\omega}^i_k$ we find

$$\partial\varepsilon^i_k/\partial t = T^i_{\cdot kjl}a^j\varepsilon^l + a^j\bar{\omega}^h \wedge (T^i_{\cdot ljh}\bar{\omega}^l_k + T^i_{\cdot klh}\bar{\omega}^l_j + T^i_{\cdot kjl}\bar{\omega}^l_h - T^l_{\cdot kjh}\bar{\omega}^i_l).$$

Consider the forms ϕ^i_{jkh} defined by

$$\phi^i_{jkh} = T^i_{\cdot ljh}\bar{\omega}^l_k + T^i_{\cdot klh}\bar{\omega}^l_j + T^i_{\cdot kjl}\bar{\omega}^l_h - T^l_{\cdot kjh}\bar{\omega}^i_l.$$

We get

$$\partial\phi^i_{jkh}/\partial t = a^m\bar{\omega}^n(T^i_{\cdot ljh}T^l_{\cdot kmn} + T^i_{\cdot klh}T^l_{\cdot jmn} + T^i_{\cdot kjl}T^l_{\cdot hmn} - T^l_{\cdot kjh}T^i_{\cdot lmn}).$$

By (3) we have $\partial\phi^i_{jkh}/\partial t=0$. Since $\phi^i_{jkh}=0$ for $t=0$, we have $\phi^i_{jkh}=0$ for all $t\in\boldsymbol{R}$. Then we have $\partial\varepsilon^i_k/\partial t=T^i_{\cdot kjl}a^j\varepsilon^l$. Therefore M is an affinely connected manifold of class c^ω. Consider the point $o=(0, \cdots, 0)$ in M. If we identify the tangent space M_o with V, the curvature tensor at o coincides with T. Since $\phi^i_{jkh}=0$ means that the covariant differential of the curvature tensor vanishes, M is an affine locally symmetric space. Then an open neighborhood of o can be extended to an affine symmetric space [**2**, p. 58].

REMARK. E. Cartan proved a similar theorem for Riemannian symmetric spaces [**1**, p. 265].

For an element A of $\mathrm{GL}(n)$ we denote by $\tilde{A}$ the Kronecker product $A\otimes A^*\otimes A^*\otimes A^*$, where A^* is the dual of A.

THEOREM. *Let K be a subgroup of* $\mathrm{GL}(n)$. *In order that there exists a simply connected affine symmetric space M such that K coincides with the linear isotropy group dH_p at some point p in M, it is necessary and sufficient that there exists a tensor T in $V\otimes V^*\otimes V^*\otimes V^*$ such that the following conditions are satisfied.*

(1) $T^i_{\cdot jkl} = -T^i_{\cdot jlk}$,

(2) $T^i_{\cdot jkl} + T^i_{\cdot klj} + T^i_{\cdot ljk} = 0$,

(3) $T^i_{\cdot hmn}T^h_{\cdot jkl} - T^h_{\cdot jmn}T^i_{\cdot hkl} - T^h_{\cdot kmn}T^i_{\cdot jhl} - T^h_{\cdot lmn}T^i_{\cdot jkh} = 0$,

(4) $K = \{A\in \mathrm{GL}(n);\ \tilde{A}T = T\}$,

where $T^i_{\cdot jkl}$ are the components of T.

PROOF. Let M be a simply connected affine symmetric space whose linear isotropy group at p $(\in M)$ is K. Since M is locally symmetric, from the Ricci identity we have

$$\begin{aligned} 0 &= \nabla_n\nabla_m R^i_{\cdot jkl} - \nabla_m\nabla_n R^i_{\cdot jkl} \\ &= R^i_{\cdot hmn}R^h_{\cdot jkl} - R^h_{\cdot jmn}R^i_{\cdot hkl} - R^h_{\cdot kmn}R^i_{\cdot jhl} - R^h_{\cdot lmn}R^i_{\cdot jkh}, \end{aligned}$$

where $R^i_{\cdot jkl}$ are the components of the curvature tensor R. Since M is a simply connected, complete affine locally symmetric space, we have $dH_p=\{A\in\mathrm{GL}(n);\ \tilde{A}R_p=R_p\}$ [5]. If we set $T=R_p$, then T satisfies (1)–(4).

Conversely if a tensor T in $V\otimes V^*\otimes V^*\otimes V^*$ satisfies (1)–(4), by the above lemma we find an affine symmetric space M and a point p in M at which $R_p=T$. Considering the universal covering manifold of M if necessary, M may be assumed to be simply connected. Then we have $dH_p=K$.

Let K be a subgroup of $\mathrm{GL}(n)$ which satisfies the conditions of the above theorem. We denote by $\mathfrak{T}_k$ the set of all tensors in $V\otimes V^*\otimes V^*\otimes V^*$ which satisfy together with K the above conditions and by $\mathfrak{S}_k$ the set of all simply connected affine symmetric spaces with linear isotropy group K. If M and M' are affinely isomorphic spaces in $\mathfrak{S}_k$, we identify M with M'. Let T and T' be tensors in $\mathfrak{T}_k$. They are said to be *equivalent* if there is A in $\mathrm{GL}(n)$ such that $T'=\tilde{A}T$. We denote by $\mathfrak{T}_{k/\sim}$ the equivalent classes of $\mathfrak{T}_k$.

COROLLARY. *There is a one-to-one correspondence between $\mathfrak{T}_{k/\sim}$ and $\mathfrak{S}_k$.*

PROOF. Let T be a tensor in $\mathfrak{T}_k$. Then by above theorem there is a space M in $\mathfrak{S}_k$ and a point p in M such that $dH_p=K$ and $R_p=T$. Let M' be a space in $\mathfrak{S}_k$ and a point p' in M' such that $dH'_{p'}=K$ and $R'_{p'}=T$. If we identify the tangent space M_p with the tangent space $M'_{p'}$ by a transformation I, I maps R_p in $R'_{p'}$. Since M and M' are simply connected, I

induces an affine isomorphism of M onto M' [3, p. 265]. Therefore a mapping $\lambda:\mathfrak{T}_k\rightarrow\mathfrak{G}_k$ is defined. We see by the above theorem that λ is surjective. Let T and T' be tensors in $\mathfrak{T}_k$ such that $\lambda(T)=\lambda(T')$. Then there are a space M in $\mathfrak{G}_k$ and points p and p' in M such that $R_p=T$, $R_{p'}=T$. Since there is an affine automorphism φ of M such that $\varphi(p)=p'$ [4, p. 223], by identifying M_p with $M_{p'}$ we have $(d\varphi)_p\in GL(n)$ and $R_{p'}=((d\varphi)^{\sim})_p R_p$. Therefore T and T' are equivalent. Conversely if T and T' are equivalent in $\mathfrak{T}_k$, clearly we have $\lambda(T)=\lambda(T')$.

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