PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

QUOTIENT AND PSEUDO-OPEN IMAGES OF SEPARABLE METRIC SPACES

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ABSTRACT. Ernest A. Michael has given a characterization of the regular quotient images of separable metric spaces. His result is generalized here to a characterization of the T_1 quotient images of separable metric spaces (which are the same as the T_1 quotient images of second countable spaces). This result is then used to characterize the Hausdorff pseudo-open images of separable metric spaces.

1. **Introduction.** A space is a *k-space* if a subset F is closed whenever $F\cap K$ is closed in K for every compact set K. A space is *sequential* if a subset F is closed whenever no sequence in F converges to a point not in F. A space is *Fréchet* if for each subset A and point $x\in \operatorname{cl} A$, there is a sequence in A converging to x. (It is easy to see that first countable spaces are Fréchet, Fréchet spaces are sequential, and sequential spaces are k-spaces.)

A *network* in a space [1] is a collection of subsets φ such that given any open subset U and $x\in U$, there is a member P of φ such that $x\in P\subset U$. A *k-network* (called a *pseudobase* by Michael [5]) is a collection of subsets φ such that given any compact subset K and any open set U containing K, there is a $P\in\varphi$ such that $K\subset P\subset U$.[1] A *cs-network* [4] is a collection of subsets φ such that given any convergent sequence $x_n\rightarrow x$ and any open set U containing x, there is a $P\in\varphi$ and a positive integer m such that $\{x\}\cup\{x_n|n\geqq m\}\subset P\subset U$. (Note that any k-network is a cs-network, which is in turn a network, and any cs-network φ which is closed under finite unions satisfies the property that given any convergent sequence $x_n\rightarrow x$ and any open U containing $\{x\}\cup\{x_n|n=1,2,\cdots\}$, there is a $P\in\varphi$ such that $\{x\}\cup\{x_n\}\subset P\subset U$.)

Received by the editors March 31, 1971.

AMS 1970 *subject classifications*. Primary 54C10, 54D55; Secondary 54D50, 54E99.

Key words and phrases. Fréchet spaces, sequential spaces, network, k-network, cs-network, $\aleph_0$-spaces, sequence-covering mappings, quotient mappings, pseudo-open mappings, separable metric spaces, second countable spaces.

[1] Paul O'Meara's definition of a k-network (Proc. Amer. Math. Soc. **29** (1971), 183–189) requires only that there be a finite union R of members of φ such that $K\subset R\subset U$. Clearly, a space with a countable k-network in this sense has one in our sense, and conversely.

THEOREM 1. *The following properties of a T_1-space X are equivalent*:

(a) *X is a sequential space with a countable k-network.*

(b) *X is a sequential space with a countable cs-network.*

(c) *X is a quotient space of a separable metric space.*

(d) *X is a quotient space of a second countable space.*

For a Hausdorff space X, these properties are equivalent to

(e) *X is a k-space with a countable k-network.*

Michael used property (e) to characterize the regular quotient images of separable metric spaces [**5**, Corollary 11.5, p. 999].[2] His proof remains valid for Hausdorff spaces.

THEOREM 2. *The following properties of a Hausdorff space X are equivalent*:

(a) *X is a Fréchet space with a countable k-network.*

(b) *X is a Fréchet space with a countable cs-network.*

(c) *X is a pseudo-open image of a separable metric space.*

(d) *X is a pseudo-open image of a second countable space.*

All mappings are assumed to be continuous and onto unless otherwise noted. Many of the ideas used in proving Theorem 1 can be found in [**5**].

The author would like to thank his teachers, Professors M.-E. Hamstrom and Ross L. Finney, for their advice and encouragement. He is also indebted to the referee, whose suggestions were extremely helpful.

2. **Proof of Theorem 1.** The following result, which is analogous to [**5**, Proposition 2.1, p. 984], will be needed.

PROPOSITION 3. *If X is a T_1 sequential space with a countable cs-network, then X has a countable k-network. In fact, any T_1 quotient space of X is a sequential space with a countable k-network*

PROOF. We prove the second statement, from which the first trivially follows.

Let $f:X\to Y$ be a quotient mapping where Y is T_1. Since every quotient space of a sequential space is sequential [**2**, Proposition 1.2, p. 109], we need only show that Y has a countable k-network. Let γ be a countable cs-network for X which is closed under finite unions. We shall show that $\varphi=\{f(B)|B\in\gamma\}$ is a k-network for Y.

Let K be a compact subset of Y contained in an open set U. Let $f(B_1)$, $f(B_2),\cdots$ be the members of φ which are contained in U, and let $A_n=f(B_1)\cup\cdots\cup f(B_n)$. Since φ clearly forms a network in Y, we know that $K\subset\bigcup\{A_n\}$. We need to show that $K\subset A_n=f(B_1\cup\cdots\cup B_n)$ for some n.

[2] Michael calls a regular space with a countable k-network an $\aleph_0$-space.

Assume the contrary and choose y_n in $K-A_n$ for each n. Since $K\subset \bigcup \{A_n\}$ and since the A_n's are increasing, the set $S=\{y_n\}$ is infinite. Since K is compact, this set has a cluster point y' in K, so that $S-\{y'\}$ is not closed in K, and hence not in U. Now $f|f^{-1}(U)$ is a quotient mapping since U is open, so $f^{-1}(S-\{y'\})$ is not closed in $f^{-1}(U)$. Since open subsets of sequential spaces are sequential [**2**, Proposition 1.9, p. 110], there exists a sequence $\{x_k\}$ in $f^{-1}(S-\{y'\})$ converging to a point x in $f^{-1}(U)-f^{-1}(S-\{y'\})$. Since Y is T_1, each $f^{-1}(y_n)$ is closed, and so the sequence $\{x_k\}$ must contain points from infinitely many of the $f^{-1}(y_n)$. Now $\{x\}\cup\{x_k\}$ is a convergent sequence contained in an open set $f^{-1}(U)$, hence we can find a member $B\in\gamma$ such that $\{x\}\cup\{x_k\}\subset B\subset f^{-1}(U)$. Then $f(B)\subset U$, so $B=B_n$ for some n. But this implies that A_n contains infinitely many of the y_n's, contradicting our choice of them above. This contradiction shows that φ is a k-network in Y.

We shall also use the following result, whose proof is due to Michael [**5**, Proposition 10.2, p. 994].

PROPOSITION 4 (MICHAEL). *A T_0-space has a countable network if and only if it is a continuous image of a separable metric space.*

The following analogues of compact-covering maps and $k(X)$ [**5**, p. 990] will be used. A mapping $f: X\to Y$ is *sequence-covering* [**6**] if, given any convergent sequence $y_n\to y_0$ in Y, there exists a convergent sequence $x_n\to x_0$ in X such that $f(x_n)=y_n$, $n=0, 1, 2, \cdots$. Given a space X, let σX [**3**, p. 52] be the set X with topology consisting of the sequentially open[3] sets of X.

The following facts are easily checked:

(1) σX is a sequential space.

(2) X is a sequential space iff $X=\sigma X$.

(3) X and σX have the same convergent sequences.

(4) If $f: X\to Y$ is continuous, then so is the induced function $f_\sigma: \sigma X\to\sigma Y$. (If U is sequentially open in Y, $f^{-1}(U)$ is sequentially open in X.)

The following result is analogous to [**5**, Lemma 11.2, p. 998] and has a straightforward proof due to Siwiec [**6**, Theorem 4.1].

PROPOSITION 5. *Any sequence-covering mapping whose range is sequential is a quotient mapping.*

Finally, we have the following analogue to [**5**, Theorem 11.4, p. 998].

[3] A subset U of X is *sequentially open* if each sequence in X converging to a point in U is eventually in U.

THEOREM 6. *The following properties of a T_1-space are equivalent*:

(a) *X has a countable cs-network.*

(b) *X is the image, under a sequence-covering mapping, of a separable metric space.*

(c) *σX is the image, under a sequence-covering quotient mapping, of a separable metric space.*

(d) *σX is a quotient space of a separable metric space.*

(e) *σX is a quotient space of a second countable space.*

The proof is analogous to Michael's:

(a)$\to$(b). Let X be a T_0-space with a countable *cs*-network φ. Let $S=\{0\}\cup\{1/n|n=1,2,\cdots\}$ with topology inherited from the real line. Let X^S be the set of mappings from S into X (i.e., the set of convergent sequences in X) with the compact-open topology. Then X^S is a T_0-space with a countable network. (Letting γ be a countable basis for S which is closed under finite unions, the set of finite intersections of sets of the form $\{u\in X^S|u(\mathrm{cl}\,Q)\subset P\}$, where $Q\in\gamma$ and $P\in\varphi$, form a network for X^S.) Hence, by Proposition 4, there exists a separable metric space M and a mapping $g:M\to X^S$.

Let $\omega:X^S\times S\to X$ be the evaluation mapping $\omega(u,s)=u(s)$. (ω is continuous since S is compact.) It is easily checked that $f=\omega\circ(g\times 1_S)$ is a sequence-covering mapping from $M\times S$ to X.

(b)$\to$(c). This follows easily from the facts about σX quoted above and from Proposition 5.

(c)$\to$(d)$\to$(e). Obvious.

(e)$\to$(a). By Proposition 3, σX has a countable *cs*-network which, by fact (3) above, is also a countable *cs*-network for X.

PROOF OF THEOREM 1. All implications except (e)$\to$(a) are either obvious or follow trivially from Theorem 6 or Proposition 3.

(e)$\to$(a). Let X be a Hausdorff k-space with a countable k-network φ. We must show that X is sequential. If $F\subset X$ is not closed, then there is a compact subset K of X such that $F\cap K$ is not closed in K. Since $\{K\cap P|P\in\varphi\}$ is a countable k-network for K, K is a metric space [**4**, Property C, p. 983]. ($\{\mathrm{int}_K(K\cap P)|P\in\varphi\}$ is a countable basis for K.) Since $F\cap K$ is not closed in K, there exists a sequence in $F\cap K$ converging to a point in $K-(F\cap K)$. Hence we have a sequence in F converging to a point not in F.

3. **Proof of Theorem 2.** We shall use the following proposition which has an easy proof due to S. P. Franklin (see [**2**, Proposition 2.3, p. 113]).

PROPOSITION 7. (i) *If $f:X\to Y$ is a quotient mapping and Y is Hausdorff and Fréchet, then f is pseudo-open.* (ii) *If X is Fréchet and $f:X\to Y$ is pseudo-open, then Y is Fréchet.*

PROOF OF THEOREM 2. (a)→(b). Obvious.

(b)→(c). If X is a Fréchet space with a countable *cs*-network, then, by Theorem 1, there exists a separable metric space M and a quotient mapping $f: M \to X$. By Proposition 7, f is pseudo-open.

(c)→(d). Obvious.

(d)→(a). If X is a pseudo-open image of a second countable space, then X is Fréchet by Proposition 7, and X has a countable k-network by Theorem 1.

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